Analysis and control of parabolic partial differential equations with application to tokamaks using sum-of-squares polynomials

Aditya Gahlawat

THÈSE

Pour obtenir le grade de

DOCTEUR DE L'UNIVERSITÉ GRENOBLE ALPES

préparée dans le cadre d'une cotutelle entre l'*Université Grenoble Alpes* et *Illinois Institute of Technology, Chicago, USA*

Spécialité : **Automatique et Productique**

Arrêté ministériel : le 6 janvier 2005 - 7 août 2006
Présentée par

Aditya GAHLAWAT

Thèse dirigée par **Mazen ALAMIR,** et **Emmanuel WITRANT**
codirigée par **Matthew PEET**

préparée au sein des **Laboratoires GIPSA-lab et Mechanical, Materials and Aerospace Engineering Department** dans **les Écoles Doctorales Electronique, Electrotechnique, Automatique et Traitement du Signal et Mechanical and Aerospace Engineering**

Analysis and Control of Parabolic Partial Differential Equations With Application to Tokamaks Using Sum-of-Square Polynomials.

Analyse et contrôle des équations aux dérivées partielles parabolique aide de polynômes somme des carrés avec une application sur les Tokamaks.

Thèse soutenue publiquement le **28 Octobre 2015**,
devant le jury composé de :

Geofferey WILLIAMSON, Président
Professeur, Illinois Institute of Technology (Chicago, USA)
Antonis PAPACHRISTODOULOU, Rapporteur
Professeur, University of Oxford (Oxford, UK)
Christian EBENBAUER, Rapporteur
Professeur, University of Stuttgart (Stuttgart, Germany)
Boris PERVAN, Examinateur
Professeur, Illinois Institute of Technology (Chicago, USA)
Bassam BAMIEH, Examinateur
Professeur, University of California at Santa Barbara (Santa Barbara, USA)
Mazen ALAMIR, Directeur de Recherche
Professeur, Universite de Grenoble, GIPSA-Lab (Grenoble, France)
Emmanuel WITRANT, Co-Directeur de Recherche
Professeur, UJF, GIPSA-Lab (Grenoble, France)
Matthew PEET, Co-Encardant de Recherche
Maître de Conferences, Illinois Institute of Technology (Chicago, USA)/Arizona State University (Tempe, USA)

TABLE OF CONTENTS

ABSTRACT

In this work we address the problems of stability analysis and controller synthesis for one dimensional linear parabolic Partial Differential Equations (PDEs). To achieve the tasks of stability analysis and controller synthesis we develop methodologies akin to the Linear Matrix Inequality (LMI) framework for Ordinary Differential Equations (ODEs). We develop a method for parabolic PDEs wherein we test the feasibility of certain LMIs using SDP to construct quadratic Lyapunov functions and controllers. The core of our approach is the construction of quadratic Lyapunov functions parametrized by positive definite operators on infinite dimensional Hilbert spaces. Unlike positive matrices, there is no single method of parametrizing the set of all positive operators on a Hilbert space. Of course, we can always parametrize a subset of positive operators, using, for example, positive scalars. However, we must ensure that the parametrization of positive operators should not be conservative. Our contribution is constructing a parametrization which has only a small amount of conservatism as indicated by our numerical results. We use Sum-of-Squares (SOS) polynomials to parametrize the set of positive, linear and bounded operators on Hilbert spaces. As the name indicates, an SOS polynomial is one which can be represented as a sum of squared polynomials. The most important property of an SOS polynomial is that it can be represented using a positive (semi)-definite matrix. This implies that even though the problem of polynomial (semi)-positivity is NP-hard, the problem of checking if polynomial is SOS (and hence (semi)-positive) can be solved using SDP. Therefore, we aim to construct quadratic Lyapunov functions parametrized by positive operators. These positive operators are in turn parametrized by SOS polynomials. This parametrization using SOS allows us to cast the feasibility problem for the existence of a quadratic Lyapunov function as the feasibility problem of LMIs. The feasibility problem of LMIs can then be addressed using SDP. In the first part of the thesis we consider stability analysis and boundary controller synthesis for a large class of parabolic PDEs. The PDEs have spatially distributed coefficients. Such PDEs are used to model processes of diffusion, convection and reaction of physical quantities in anisotropic media. We consider boundary controller synthesis for both the state feedback case and the output feedback case (using an observer design). In the second part of thesis we design distributed controllers for the regulation of poloidal magnetic flux in a tokamak (a thermonuclear fusion device). First, we design the controllers to regulate the magnetic field line pitch (the safety factor). The regulation of the safety factor profile is important to suppress the magnetohydrodynamic instabilities in a tokamak. Then, we design controllers to maximize the internally generated bootstrap current density. An increased proportion of bootstrap current would lead to a reduction in the external energy requirements for the operation of a tokamak.

CHAPTER 1

RÉSUMÉ EN FRANÇAIS

Dans ce travail, nous considérons l'analyse et le contrôleur et la synthèse d'observateur pour les Équations Différentielles Partielles (EDP) paraboliques en utilisant polynômes Somme des carrés (SOS). Dans les Chapitres 5-7 nous considérons une classe générale des EDP paraboliques. Considérant que, dans les Chapitres 8-9 nous considérons la PDE régissant l'évolution du flux magnétique poloïdal dans un Tokamak.

Dans le Chapitre 5 nous analysons la stabilité de

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t),$$

avec des conditions aux limites

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{et} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = 0.$$

Ici a, b et c sont des fonctions polynomiales de $x \in [0,1]$. En outre,

$$|\nu_1| + |\nu_2| > 0 \quad \text{et} \quad |\rho_1| + |\rho_2| > 0. \tag{1.1}$$

Différentes valeurs de ces scalaires peuvent être utilisés pour représenter Dirichlet, Neumann, Robin ou des conditions aux limites mixtes.

Nous établissons la stabilité exponentielle en construisant des fonctions de Lyapunov de la forme $V(w(\cdot,t)) = \langle w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle$, où

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1), \tag{1.2}$$

où les polynômes M, K_1 et K_2 sont paramétrés par des matrices positives. Les résultats des expériences numériques présentées prouver que la méthode présentée a une quantité négligeable de conservatisme.

Dans le Chapitre 6 nous construisons de façon exponentielle stabiliser contrôleurs basés retour d'état pour

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t),$$

avec des conditions aux limites

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{et} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t).$$

Ici $u(t) \in \mathbb{R}$ est l'entrée de commande. Utilisation des fonctions de Lyapunov de la forme $V(w(\cdot,t)) = \langle w(\cdot,t), \mathcal{P}^{-1}w(\cdot,t)\rangle$, où $\mathcal{P}$ est de la forme donnée dans l'Équation (10.2), nous synthétisons contrôleurs $\mathcal{F} : H^2(0,1) \to \mathbb{R}$ de telle sorte que si la commande est donnée par

$$u(t) = \mathcal{F}w(\cdot,t),$$

alors le système est exponentiellement stable. Des expériences numériques indiquent que la méthode est très efficace dans des systèmes qui sont contrôlables dans un certain sens approprié de stabilisation. En outre, nous étendons la méthodologie de construction L_2 contrôleurs de limites optimales qui minimisent l'effet d'une entrée décentralisée exogène sur l'état du système.

Dans le Chapitre 7 nous construisons de façon exponentielle estimation observateurs d'état pour

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t),$$

avec des conditions aux limites

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{et} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t).$$

Nous supposons que la mesure limite (sortir) de la forme

$$y(t) = \mu_1 w(1,t) + \mu_2 w_x(1,t),$$

est disponible. L'objectif est d'estimer l'état w le système à l'aide de la sortie frontière y. Pour ce faire, nous concevons des observateurs de Luenberger de la forme

$$\hat{w}_t(x,t) = a(x)\hat{w}_{xx}(x,t) + b(x)\hat{w}_x(x,t) + c(x)\hat{w}(x,t) + p(x,t),$$

avec des conditions aux limites

$$\hat{w}_1 w(0,t) + \nu_2 \hat{w}_x(0,t) = 0 \quad \text{et} \quad \rho_1 \hat{w}(1,t) + \rho_2 \hat{w}_x(1,t) = u(t) + q(t).$$

Ici $p(x,t)$ et $q(t)$ sont les entrées d'observateurs.

En construisant des fonctions de Lyapunov de la forme

$$V((\hat{w} - w)(\cdot,t)) = \langle (\hat{w} - w)(\cdot,t), \mathcal{P}(\hat{w} - w)(\cdot,t) \rangle\,,$$

nous construisons opérateur $\mathcal{O} : \mathbb{R} \to L_2(0,1)$ et scalaire O de telle sorte que si

$$p(x,t) = \left(\mathcal{O}(\hat{y}(t) - y(t))\right)(x) \quad \text{et} \quad q(t) = O(\hat{y}(t) - y(t)),$$

où $\hat{y}(t) = \mu_1 \hat{w}(1,t) + \mu_2 \hat{w}_x(1,t)$, puis $\hat{w} \to w$ exponentiellement vite. En outre, nous montrons que les observateurs conçues peuvent être couplés à des contrôleurs conçus dans le Chapitre 6 à construire de façon exponentielle en fonction de stabilisation observateurs contrôleurs de limites. Les résultats numériques indiquent que la méthode proposée est efficace dans la construction de rétroaction de sortie contrôleurs.

Dans les Chapitres 8-9 on considère le gradient de flux magnétique poloïdal la $Z = \psi_x$ dont l'évolution est régie par

$$\frac{\partial Z}{\partial t}(x,t) = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_\parallel(x,t)}{x}\frac{\partial}{\partial x}\left(xZ(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_\parallel(x,t) j_{lh}(x,t) + j_{bs}(x,t)\right),$$

avec des conditions aux limites

$$Z(0,t) = 0 \quad \text{et} \quad Z(1,t) = -R_0\mu_0 I_p(t)/2\pi,$$

où

$\eta_\parallel$ = résistivité parallèle,

j_{lh} = hybride basse densité de courant (LHCD),

j_{bs} = densité de courant d'amorçage,

I_p = courant de plasma totale, et

μ_0 = la perméabilité de l'espace libre.

Dans le Chapitre 8 nous réglementons le terrain des lignes de champ magnétique, également connu sous le profil de facteur de sécurité, ou la q-profil en utilisant j_{lh} que l'entrée de commande. Depuis

$$q \propto \frac{1}{Z},$$

nous réglementons le Z-profil. Nous accomplissons cette tâche en utilisant une fonction de Lyapunov de partir

$$V(Z(\cdot,t)) = \int_0^1 x^2(1-x)M(x)^{-1}Z(x,t)^2 dx,$$

où $M(x)$ est un polynôme strictement positif et

$$j_{lh}(x,t) = K_1(x)Z(x,t) + \frac{\partial}{\partial x}\left(K_2(x)Z(x,t)\right),$$

où K_1 et K_2 sont des fonctions rationnelles.

Dans le Chapitre 9 nous maximisons la norme de la densité de courant bootstrap j_{bs}. Depuis

$$j_{bs} \propto \frac{1}{Z},$$

nous minimisons la norme de la Z-profil. Nous utilisons une fonction de la forme Lypaunov

$$V(Z(\cdot,t)) = \int_0^1 x^2 M(x)^{-1}Z(x,t)^2 dx,$$

où $M(x)$ est un polynôme strictement positif et

$$j_{lh}(x,t) = K_1(x)Z(x,t),$$

où K_1 est une fonction rationnelle. De plus, nous présentons une heuristique de telle sorte que les contraintes de forme sur l'entrée de commande j_{lh} sont respectés.

CHAPTER 2
INTRODUCTION

In the year 2011, fossil fuel energy accounted for 83% of the total global consumption. Despite the fact that renewable energy and nuclear fission power are the world's fastest growing energy sources, fossil fuels will continue to supply almost 80% of the global demand through 2040 [1]. It is because of this dependence on fossil fuels that the total carbon emissions are expected to rise by 29% during the same time period [2]. Moreover, before the end of the 21^{st} century, an energy shortfall is expected to occur if only the present energy sources like fossil fuels, hydro and nuclear fission are used [3]. Although renewable energy sources like solar, wind and geothermal energy are safe and cause a minimal environmental impact (green house gases emission and ecological damage), they do not posses the desired energy production density (rate of energy produced divided by the area of the land required to produce it). Thus, an energy source is required which has abundant fuel, possesses high energy density, causes a minimal environmental impact and is safe.

A possible energy source that satisfies all the requirements highlighted in the previous paragraph is nuclear fusion [4]. Nuclear fusion is the process in which two nuclei fuse to form a single nucleus and possibly additional neutrons and protons. Consider the reaction

$$H^2 + H^3 \to He^4 + n,$$

where H^2 denotes a Deuterium nucleus (one proton and one neutron), H^3 is the Tritium nucleus (one proton and two neutrons), He^4 is the Helium nucleus (two protons and two neutrons) and n is a neutron. In order for the Deuterium and Tritium particles to overcome the electrostatic force of repulsion and fuse, they must possess significant energy. This energy may be provided by heating up the Deuterium-Tritium gas to a temperature of a 100 million degrees Celsius. At a temperature of 100 million degrees Celsius, the Deuterium-Tritium gas is in a completely ionized state, also known as a plasma. Since the Deuterium-Tritium plasma has free electrons and ions, the plasma can be confined by a magnetic field. This is because a charged particle moving through a magnetic field experiences a force (Lorentz force) that causes it to gyrate about the magnetic field lines [5]. A *tokamak* is a toroidal vessel that uses magnetic fields to confine plasmas. A tokamak is equipped with current carrying coils arranged around the toroid (see Fig. 2.1). These current carrying coils create a magnetic field B_T which lies in the toroidal plane. Additionally, a tokamak has a current carrying core which is charged before the initiation of the fusion and then is commanded to discharge. This discharge generates a varying magnetic field around the plasma. Since the plasma is a conductor, a current I_p is generated described by Faraday's laws of induction. The plasma current I_p generates a magnetic field B_P in a plane normal to the toroidal plane. The combination of B_P and B_T produces a helical magnetic field that confines the plasma [6], [7].

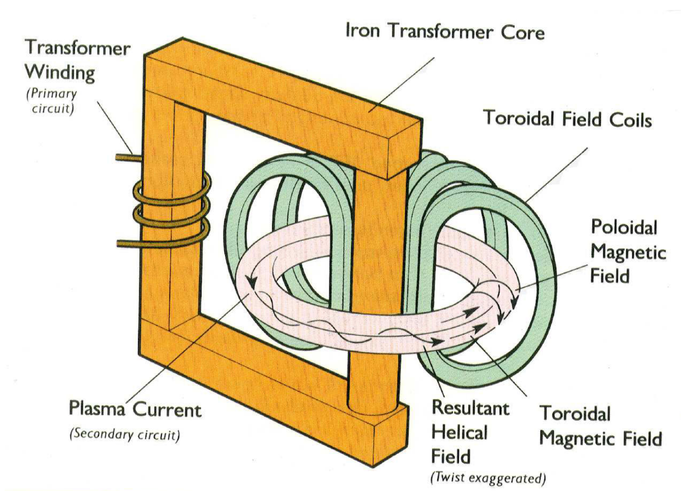


Figure 2.1. Schematic of a tokamak.

The word 'tokamak' is derived from the Russian for 'toroidal chamber' and 'magnetic coil'. The T-1 tokamak, built in the former USSR, for the first time since research in fusion devices began, achieved temperature and confinement times required for the initiation of fusion [8]. It was soon realized that an improvement in the plasma confinement time could be achieved by increasing the plasma minor radius [6]. Thus, many countries undertook the project of designing and building larger tokamaks. The largest of these was the Joint European Torus (JET) tokamak [9]. The JET tokamak, and others like the Tore Supra [10], have been used for a better understanding of tokamak plasma physics and simulating conditions for future tokamaks. One such future tokamak is the *iter* tokamak [4]. Iter is a large tokamak currently under construction in southern France and is jointly funded by China, the European Union, India, Japan, South Korea, Russia and the United States. The goal of iter is to demonstrate the technology for electricity generation using thermonuclear fusion.

The plasma in a tokamak suffers from various instabilities. For example, an important instability which occurs at the plasma center is the *sawtooth instability* [11]. The sawtooth instability causes the temperature and pressure at the center of the plasma to rise and crash in a periodic fashion. The crash in the temperature and pressure results from a fast outward transport of particles and energy from the center. This transport removes the energetic particles from the plasma center which are required for the fusion to continue. Additionally, large sawteeth can trigger other instabilities in the plasma [12].

Another example of a plasma instability is the *Neoclassical Tearing Mode* (NTM) instability. The magnetic field confining a tokamak plasma can be thought of as nested iso-flux toroids. The NTM instability occurs when the iso-flux surfaces tear and rejoin to form structures known as *magnetic islands* [13]. The presence of the magnetic islands adversely affects the energy confinement and reduces the plasma pressure. For example, if the NTM instabilities were allowed to grow in the iter tokamak, the magnetic islands would cover a third of the total plasma volume and reduce the fusion power production by a factor of four [14].

The suppression of such plasma instabilities and various others require feedback control to achieve desired plasma properties in a tokamak. Feedback control can be used to improve the safety and efficiency of tokamaks. A few examples of feedback control applications in a tokamak include, plasma shape [15], [16], safety factor [17], [18] and plasma pressure and current [19], [20]. Moreover, the *iter* tokamak [4] will be operating under the Advanced Tokamak (AT) regime [21]. The AT regime requires plasma shapes with a high degree of accuracy, high plasma pressures, increased plasma confinement efficiency and a reduction in the dependence on external energy input. Due to the importance of feedback control, large tokamaks like JET [22] and DIII-D [23] have ongoing programs dedicated to the design and validation of controllers for the AT regime [24], [25], [26], [27].

A tokamak plasma interacts with currents, magnetic fields and forces exerted on and by it. In order to quantitatively predict the behavior of tokamak plasmas, mathematical models are required. One way is to use Magneto-Hydro-Dynamics (MHD) models. MHD is a branch of physics that studies the behavior of plasma under the effects of electric and magnetic fields [28]. A sub-branch of MHD is ideal MHD [29], wherein we make the assumption that the plasma has zero resistivity. However, ideal MHD is sufficiently accurate in predicting certain plasma instabilities and its models can be used to construct plasma evolution equations for control design [7]. Ideal MHD models of plasmas are derived using Maxwell's equations and conservation of mass, momentum and energy [30]. Recall, Maxwell's equations are a set of four equations (Gauss' law for electricity, Gauss' law for magnetism, Faraday's laws of induction and Ampere's law) which describe how electric and magnetic fields interact, propagate, influence and get influenced by objects.

Maxwell's equations, and hence models of MHD, are described by Partial Differential Equations (PDEs). To understand what a PDE is, consider n variables

$x_1, \cdots, x_n$, $x_j \in \Omega \subset \mathbb{R}$, $j \in \{1, \cdots, n\}$, and quantity $w(x_1, \cdots, x_n)$, $w : \Omega \times \cdots \times \Omega \rightarrow \mathbb{R}$. A general one dimensional PDE model is of the form [31]:

$$F\left(x_1, \cdots, x_n, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}, \frac{\partial^2 w}{\partial x_1 x_2}, \cdots, \frac{\partial^{(i)} w}{\partial x_1^{(i)}}, \cdots\right) = 0, \tag{2.1}$$

where $F : \Omega \times \cdots \times \Omega \times \mathbb{R} \times \cdots \times \mathbb{R} \rightarrow \mathbb{R}$, $\frac{\partial w}{\partial x_j}$, $j \in \{1, \cdots, n\}$, denote the partial derivative of $w(x_1, \cdots, x_n)$ with respect to x_j and $i \in \mathbb{N}$. In this work, we consider PDEs of the form

$$w_t(x, t) = a(x) w_{xx}(x, t) + b(x) w_x(x, t) + c(x) w(x, t), \tag{2.2}$$

where $x \in [0, 1]$, $t \geq 0$ and a, b and c are continuous functions of the independent variable x. Such types of PDEs are known as *second order parabolic PDEs*. Parabolic PDEs are used to model processes such as diffusion, transport and reaction. The choice of such PDEs is partially motivated by the models employed for the evolution of plasma parameters in a tokamak. However, such models have coefficients which are both space and dependent. Since we are interested in the steady state operation of tokamaks, i.e., holding the plasms stable to some equilibrium, we drop the temporal dependencies of the coefficients and consider the simplified models. Such models may be used to depict the evolution of plasma parameters at time scales much slower than the MHD modes.

The first question to be asked of a parabolic PDE, or in fact any type of PDE, is if it is well-posed. A parabolic PDE is well-posed if the PDE has a unique solution. The definition of a solution of a PDE is non-trivial [31], [32], [33], [34]. To keep the introduction simple, we will use the ‘classical definition of the solution’. Rigorous definitions of solutions of PDEs and their types will be presented in subsequent chapters. Consider the parabolic PDE given in Equation (2.2). Intuitively, it can be seen that a solution w to this second order PDE is one which is atleast twice continuously differentiable in x and continuously differentiable in t, such that all the derivatives are well-defined, and w satisfies the equation. These requirements lead to the definition of a classical solution.

Definition 2.1. *[31] For the PDE given in* (2.2)*, a function which is at least twice continuously differentiable in x, continuously differentiable in t and satisfies the PDE is known as a solution. If in addition, the solution is unique, it is defined as a classical solution.*

Since the concept of classical solution is the easiest to understand, we will use it throughout the introduction.

We now consider the problem of stability analysis. To this end, we will start by defining a set of real valued functions known as $L_2(\Omega)$, $\Omega \subset \mathbb{R}$, given as

$$L_2(\Omega) := \{f : \Omega \to \mathbb{R} : \|f\|_{L_2} = \left(\int_\Omega f^2(x)dx\right)^{\frac{1}{2}} < \infty\}. \tag{2.3}$$

The set $L_2(\Omega)$ is widely used in the analysis of PDEs and thus, we will use it in the subsequent discussion. The functional $\|\cdot\|_{L_2} : L_2(\Omega) \to \mathbb{R}$ is known as the norm on the set $L_2(\Omega)$. The definition and properties of norms can be found in [35]. For any $f \in L_2(\Omega)$, the norm $\|f\|_{L_2}$ formalizes the concept of 'the size' of f. Similarly, for $f, g \in L_2(\Omega)$, the norm $\|f - g\|_{L_2}$ quantifies the 'closeness' of f and g. With the understanding of L_2 and its norm $\|\cdot\|_{L_2}$, we can now define the stability of solutions of PDEs. In particular, we are interested in exponential stability defined as following.

Definition 2.2. *The PDE given in Equation* (2.2) *is* ***exponentially stable in the sense of $\mathbf{L_2(\Omega)}$*** *if there exist scalars $M > 0$ and $\alpha > 0$ such that*

$$\|w(\cdot, t)\|_{L_2} \leq Me^{-\alpha t} \quad \textit{for all} \quad t > 0.$$

As an example, consider the stability of the one dimensional heat conducting rod whose temperature $w(x, t)$, $x \in [0, 1]$, $t > 0$, is governed by the parabolic PDE

$$w_t(x, t) = \kappa w_{xx}(x, t),$$

where $\kappa > 0$ is the thermal conductivity of the rod. Additionally, suppose that the temperature of the rod is zero at both ends. This results in the following boundary conditions

$$w(0,t) = 0 \quad \text{and} \quad w(1,t) = 0, \quad \text{for all} \quad t > 0.$$

The solution to this PDE is given by [36]:

$$w(x,t) = 2\kappa \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t} \sin(\pi n x) \int_0^1 \sin(\pi n z) w(z,0) dz.$$

It is easy to show that

$$\|w(\cdot,t)\| \leq M e^{-\alpha t}, \quad \text{for all} \quad t > 0,$$

where

$$M = 2\kappa \left(\int_0^1 \sum_{n=1}^{\infty} \sin^2(\pi n x) \int_0^1 \sin^2(\pi n z) w^2(z,0) dz dx \right)^{\frac{1}{2}} \quad \text{and} \quad \alpha = \pi^2.$$

Thus, using Definition 2.2 it can be seen that the heat equation is exponentially stable.

Consider the following extension of the PDE given in Equation (2.2):

$$w_t(x,t) = a(x) w_{xx}(x,t) + b(x) w_x(x,t) + c(x) w(x,t) + d(x) u_1(x,t), \tag{2.4}$$

with boundary conditions

$$w(0,t) = 0 \text{ and } w_x(1,t) = \beta u_2(t),$$

where a, b, c and d are known continuously differentiable coefficients, β is a known scalar and $w(\cdot,t) \in L_2(0,1)$. The functions $u_1 : (0,1) \times (0,\infty) \to \mathbb{R}$ and $u_2 : (0,\infty) \to \mathbb{R}$, which appear in the PDE in addition to the dependent variables and the unknown function w, are known as **inputs**. The distributed function of x, $u_1(x,t)$, is known as a **distributed input**. The function $u_2(t)$ which appears in the boundary conditions is known as a **boundary input**. The case when $d(x) = 0$ is an example of the system with only boundary input. Similarly, the system only has distributed input when $\beta = 0$.

For PDEs with input, we consider exponential stabilization and regulation defined as follows:

Definition 2.3 (Stabilization)**.** *For the PDE 2.4, the* ***stabilization problem*** *is:*

$$\begin{aligned} \textit{Find:} \quad & u_1(x,t) \text{ and } u_2(t) \\ \textit{such that:} \quad & \text{there exist } M, \alpha > 0 \text{ with } \|w(\cdot,t)\| \le Me^{-\alpha t}, \quad t \ge 0. \end{aligned}$$

Definition 2.4 (Regulation)**.** *Given a function $v(x)$, the* ***regulation problem*** *is:*

$$\begin{aligned} \textit{Find:} \quad & u_1(x,t) \text{ and } u_2(t) \\ \textit{such that:} \quad & \text{there exist } M, \alpha > 0 \text{ with } \|w(\cdot,t) - v(\cdot)\| \le Me^{-\alpha t}, \quad t \ge 0. \end{aligned}$$

Some examples of stabilization and regulation of parabolic PDEs can be found in [37], [38], [39].

Consider the autonomous (without inputs) parabolic PDE for $x \in [0,1]$ and $t \in (0,\infty)$,

$$\begin{aligned} & w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t), \\ & w(0,t) = 0, \quad w_x(1,t) = 0, \quad y_1(x,t) = d(x)w(x,t), \quad y_2(t) = \gamma w(1,t), \end{aligned} \tag{2.5}$$

where a, b, c, d are known continuously differentiable functions and γ is a known scalar. Assume that $y_1(x,t)$ and $y_2(t)$ are known functions. These known functions which provide a complete or partial knowledge of w are known as the **outputs**. When the output provides the knowledge of w over a non-zero Lebesgue measure subset of $[0,1]$, it is known as **distributed output**. When the output provides the knowledge of w over the boundary of the set $[0,1]$, it is known as **boundary output**. In Equation (2.5), $y_1(x,t)$ is the distributed output and $y_2(t)$ is the boundary output.

Since in most cases, the outputs provide only a partial knowledge of the solution, it is desirable to use the outputs to estimate the complete solution of the PDEs. The estimates may be used for the design of stabilizing control laws, for example. To estimate the solution, an artificial PDE is constructed that uses the output of the actual PDE as its input. This artificial PDE whose output is the estimate of the solution of the actual PDE is known as the *observer*. For the PDE given by Equation (2.5), an observer of the following type, also known as a *Luenberger Observer*, can be designed

$$\begin{aligned}
&\hat{w}_t(x,t) = a(x)\hat{w}_{xx}(x,t) + b(x)\hat{w}_x(x,t) + c(x)\hat{w}(x,t) + \hat{d}(x)\left(\hat{y}_1(x,t) - y_1(x,t)\right), \\
&\hat{w}(0,t) = 0, \quad \hat{w}_x(1,t) = \hat{\gamma}(\hat{y}_2(t) - y_2(t)),
\end{aligned} \tag{2.6}$$

where $\hat{y}_1(x,t) = d(x)\hat{w}(x,t)$ and $\hat{y}_2(t) = \gamma\hat{w}(1,t)$. The search for the unknown coefficients $\hat{d}$ and $\hat{\gamma}$ is known as the *observer synthesis* problem and can be stated as follows.

Definition 2.5 (Observer synthesis)**.** *Given the linear second order PDE 2.5 with outputs y_1 and y_2, the* ***observer synthesis*** *problem is*

$$\begin{aligned}
\textit{Find:} \quad & \hat{d}(x) \quad \textit{and} \quad \hat{\gamma} \textit{ for the System 2.6} \\
\textit{such that:} \quad & \textit{there exist } M, \alpha > 0 \textit{ with} \\
& \|w(\cdot,t) - \hat{w}(\cdot,t)\| \le Me^{-\alpha t}, \quad t \ge 0.
\end{aligned}$$

A few examples of observer synthesis for parabolic PDEs can be found in [40], [41], [42].

The stabilization problem can be restated as a question of feasibility. A general optimization problem is of the form

$$\text{Minimize}_{x_i \in \mathbb{R}}: \quad f(x_1, \cdots, x_n)$$

$$\text{subject to}: \quad |g(x_1, \cdots, x_n)| \leq b \quad \text{and}$$
$$|x_1| \leq c, \cdots, |x_n| \leq c,$$

where $f, g : \mathbb{R}^n \to \mathbb{R}$, $b, c > 0$. The related feasibility problem would be to find $x_i \in \mathbb{R}$, $i \in \{1, \cdots, n\}$, which satisfy the constraints of the optimization problem.

An important type of optimization is *convex optimization* [43].

Definition 2.6 (Convex function)**.** *A real valued function $f : \mathbb{R}^n \to \mathbb{R}$ is **convex** if*

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \tag{2.7}$$

for all $x, y \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$.

This convexity condition means that a line joining any two points on the function always lies on or above the function. For convex functions, we define the following class of optimization problems.

Definition 2.7 (Convex optimization problem)**.** *A **convex optimization** problem is of the form*

$$\begin{aligned} \textit{Minimize}_{x \in \mathbb{R}^n} : &\quad f_0(x) \\ \textit{subject to} : &\quad f_i(x) \leq c_i, \quad c_i \in \mathbb{R}, \quad i \in \{1, \cdots, m\}, \end{aligned}$$

where the functions f_0 and f_i are all convex.

Constrained optimization problems, for most cases, cannot be solved analytically. However, convex optimization problems can be efficiently solved algorithmically [44]. An important class of convex optimization is a *Semi-Definite Programming* (SDP).

Definition 2.8. *An **SDP problem** is an optimization problem of the form*

$$\textit{Minimize}_{x \in \mathbb{R}^n} : \quad c^T x$$

$$\textit{subject to}: \quad F_0 + \sum_{i=1}^{n} x_i F_i \leq 0 \quad \textit{and}$$

$$Ax = b,$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, $A \in \mathbb{R}^{k \times n}$ *and symmetric matrices* $F_i \in \mathbb{S}^m$ *are given.*

We use SDP to perform stability analysis, stabilization and observer synthesis for parabolic PDEs. To explain how we accomplish these tasks, we will change the way we represent parabolic PDEs. Consider the following equation

$$\begin{aligned} &w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t), \\ &w(0,t) = 0, \quad w_x(1,t) = 0, \end{aligned} \tag{2.8}$$

where $t \in (0, \infty)$, $x \in (0,1)$ and the coefficients a, b and c are continuously differentiable. Consider the mapping

$$\mathbf{w} : (0, \infty) \to L_2(0,1)$$

defined by

$$(\mathbf{w}(t))(x) = w(x,t) \quad (x \in (0,1), \quad t \in (0,\infty)).$$

Additionally, let

$$\mathcal{A}z(x) = a(x)z_{xx}(x) + b(x)z_x(x) + c(x)z(x), \text{ for } z \in \mathcal{D}_{\mathcal{A}},$$

where

$$\begin{aligned} \mathcal{D}_{\mathcal{A}} = \{&z \in L_2(0,1) : z, z_x \text{ are absolutely continuous }, z_{xx} \in L_2(0,1), \\ &z(0) = 0 \text{ and } z_x(1) = 0\}. \end{aligned}$$

With these definitions, Equation (2.8) can be written as

$$\dot{\mathbf{w}}(t) = \mathcal{A}\mathbf{w}(t), \quad \mathbf{w}(t) \in \mathcal{D}_{\mathcal{A}}. \tag{2.9}$$

With this representation, we can provide Lyapunov inequalities for linear PDEs. We begin by providing the following definitions

Definition 2.9. *A mapping $\mathcal{P} : L_2(\Omega) \to L_2(\Omega)$, $\Omega \subset \mathbb{R}$, is a* ***bounded linear operator*** *if for all $y, z \in L_2(\Omega)$ and $\omega \in \mathbb{R}$ there exists a scalar $\xi > 0$ such that*

$$\mathcal{P}(y+z) = \mathcal{P}y + \mathcal{P}z, \quad \mathcal{P}(\omega y) = \omega \mathcal{P}y, \quad \|\mathcal{P}y\|_{L_2} \leq \xi \|y\|_{L_2}.$$

The set of all such operators is denoted by $\mathcal{L}(L_2(\Omega))$.

Definition 2.10. *An operator $\mathcal{P} \in \mathcal{L}(L_2(\Omega))$ is* ***positive*** *if for all $y, z \in L_2(\Omega)$, there exists a positive scalar ζ such that*

$$\langle \mathcal{P}y, z \rangle_{L_2} = \langle y, \mathcal{P}z \rangle_{L_2}, \quad \langle Py, y \rangle_{L_2} \geq \zeta \|y\|^2_{L_2}.$$

With these definitions, we now provide the Lyapunov inequalities for the stability analysis of linear PDEs.

Theorem 2.11. *[45] A given linear PDE*

$$\dot{\mathbf{w}}(t) = \mathcal{A}\mathbf{w}(t)$$

is exponentially stable if and only if there exists a $\mathcal{P} \in \mathcal{L}(L_2(\Omega))$ and a scalar $\alpha > 0$ such that

$$\begin{aligned}
&\langle \mathcal{P}z, z \rangle_{L_2} \geq 0, \text{ and} \\
&\langle \mathcal{A}z, \mathcal{P}z \rangle_{L_2} + \langle \mathcal{P}z, \mathcal{A}z \rangle_{L_2} \leq -\alpha \langle z, z \rangle_{L_2}, \text{ for all } z \in \mathcal{D}_{\mathcal{A}}.
\end{aligned}$$

There is no single method that can search over the set of positive operators to find a solution of the Lyapunov inequalities for PDEs given in Theorem 2.11. We use *Sum-of-Squares* (SOS) polynomials to parametrize positive operators. By definition, an SOS polynomial is non-negative. Moreover, an SOS polynomial can be represented using a PSD matrix [46]. Thus, a positive operator parametrized by an SOS polynomial can be represented by a PSD matrix. This implies that the search for a solution of the Lyapunov inequalities for linear PDEs can be performed over the set of PSD matrices. Hence, the problem of searching for a positive operator satisfying the Lyapunov inequalities can be cast as an SDP feasibility problem. The parametrization of operators using SOS polynomials and the setup of the Lyapunov inequalities as SDPs are discussed in subsequent chapters. Similarly, the search for controllers and observers can be cast as SDP feasibility problems.

The gradient of poloidal magnetic flux is an important physical quantity for the safe and efficient operation of tokamaks since it is related to the magnetic field line pitch, known as the safety factor profile, and the self-generated bootstrap current in the plasma. The dynamics of the gradient of poloidal magnetic flux are governed by a parabolic PDE [47]. The control is exercised using distributed input. The actuators available to administer the input use electromagnetic waves at the cyclotron frequency of electrons and ions. Unfortunately, the control input is shape constrained and the best estimates for the allowable control inputs are empirical. Nevertheless, we are able to apply similar methodologies which we develop for a general class of parabolic PDEs.

2.1 Outline

This thesis is organized as follows:

- Chapter 3 presents a brief introduction to convex optimization, Semi-Definite Programming (SDP) and Sum-of-Squares (SOS) polynomials.
- Chapter 4 presents the model of the poloidal magnetic flux in a tokamak which is utilized in Chapters 8 and 9.
- Chapter 5 presents a methodology to analyze the stability of a large class of one-dimensional linear PDEs. We use positive operators on Hilbert spaces parameterized by SOS polynomials.

- Chapter 6 presents a methodology to synthesize exponentially stabilizing boundary controllers for the class of PDEs considered in Chapter 5. We use SOS polynomials and positive operators to construct quadratic Lyapunov function and controller gains. An extension of this method is also provided wherein the synthesized controllers are shown to be L_2-optimal in the presence of an exogenous distributed input.

- Chapter 7 presents a similar methodology to the one constructed in Chapter 6 to synthesize boundary observers which utilize only the boundary measurement of the state of the plant. It is then shown that these observers may be coupled to the controllers designed previously to produce exponentially stabilizing output feedback boundary controllers.

- Chapter 8 provides a control methodology of regulating the safety factor in a profile. This is accomplished by regulating the poloidal magnetic flux using a simplified version of the model in Chapter 4 and applying a simplified version of the methodology developed in Chapter 6. A numerical simulation for the Tore Supra tokamak is also provided.

- Chapter 9 provides a control methodology for the maximization of the bootstrap current density in a tokamak. Using a simplified version of the methodology considered in Chapter 6, we develop the control method using the poloidal magnetic flux model with uncertain spatio-temporal coefficients. A numerical simulation for the Tore Supra tokamak is also provided.

2.2 Notation

The following notation and definitions are used throughout the Thesis. For a detailed discussion of the definitions used, refer to [35], [48] or the appendix of [45].

Function Spaces The following are defined for $-\infty < a < b < \infty$

- The ***Hilbert space*** $L_2(a,b)$ is defined as

$$L_2(a,b) := \{f : (a,b) \to \mathbb{R} : \|f\|_{L_2} = \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}} < \infty\}.$$

- For any Hilbert space X and scalar $0 < \tau < \infty$, we denote

$$L_2([0,\tau];X) := \{f : [0,\tau] \to X : \|f\|_{L_2([0,\tau];X)} = \left(\int_0^\tau]\|f(t)\|_{L_2}^2 dt\right)^{\frac{1}{2}} < \infty.\}.$$

Similarly, a function $f \in L_2^{loc}([0,\infty];X)$ if $f \in L_2([0,\tau];X)$ for every $\tau \geq 0$.

- For any $f, g \in L_2(a,b)$, $\langle f, g\rangle_{L_2} = \int_a^b f(x)g(x)dx$.

- Unless otherwise indicated, $\langle \cdot, \cdot \rangle$ denotes the inner product on L_2 and $\|\cdot\| = \|\cdot\|_{L_2}$ denotes the norm induced by the inner product.

- A function $f : (a,b) \to \mathbb{R}$ is ***absolutely continuous*** if for any integer N and any sequence $t_1, \cdots, t_N$, we have $\sum_{k=1}^{N-1} |x(t_k) - x(t_{k+1})| \to 0$ whenever $\sum_{k=1}^{N-1} |t_k - t_{k+1}| \to 0$.

- The ***Sobolev space*** $H^m(a,b)$ is defined as

$$H^m(a,b) := \{f \in L_2(a,b) : f, \cdots, \frac{d^{m-1}f}{dx^{m-1}} \text{ are absolutely continuous on } (a,b) \text{ with } \frac{d^m f}{dx^m} \in L_2(a,b)\}.$$

- For any $f, g \in H^m(a,b)$,

$$\langle f, g\rangle_{H^m} = \sum_{n=0}^{m} \left\langle \frac{d^n f}{dx^n}, \frac{d^n g}{dx^n} \right\rangle_{L_2}.$$

- The set of **n** ***times continuously differentiable functions*** is defined as

$$C^n(a,b) := \{f : (a,b) \to \mathbb{R} : f, \cdots, \frac{d^n f}{dx^n} \text{ exist and are continuous}\}.$$

- The set of ***smooth functions*** is defined as

$$C^\infty(a,b) := \{f : (a,b) \to \mathbb{R} : f \in C^n(a,b) \text{ for any } n \in \mathbb{N}\}.$$

- For a set X and scalar $0 < \tau < \infty$, we denote

$$C^n([0,\tau];X) := \{f : [0,\tau] \to X : f \text{ is } n\text{-times continuously}$$
$$\text{differentiable on}[0,\tau]\}.$$

  Similarly, a function $f \in C^n_{loc}([0,\infty];X)$ if $f \in C^n([0,\tau];X)$ for every $\tau \geq 0$.

- ***The direct sum of* n *Hilbert spaces* X** is denoted by X^n.

Operators on Hilbert Spaces The following are defined for any two Hilbert spaces X and Y with respective norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ and inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$.

- A mapping $\mathcal{P} : X \to Y$ is a ***linear operator*** is for all $f, g \in X$ and scalars β, it holds that $\mathcal{P}(f+g) = \mathcal{P}f + \mathcal{P}g$ and $\mathcal{P}(\beta f) = \beta\mathcal{P}f$.

- A linear operator $\mathcal{P} : X \to Y$ is a ***bounded linear operator*** if for all $f \in X$, there exists a scalar $\omega > 0$ such that $\|\mathcal{P}f\|_Y \leq \omega\|f\|_X$.

- We say that $\mathcal{P} \in \mathcal{L}(X,Y)$ if $\mathcal{P} : X \to Y$ is a bounded linear operator. Similarly, we denote by $\mathcal{L}(X)$ the set of all bounded linear operator mapping the elements of X back to itself.

- For $\mathcal{P} \in \mathcal{L}(X,Y)$, we define

$$\|P\|_{\mathcal{L}(X,Y)} = \sup_{f \in X, \|f\|_X = 1} = \|\mathcal{P}f\|_Y.$$

- For any $\mathcal{P} \in \mathcal{L}(X,Y)$, there exists a unique $\mathcal{P}^\star \in \mathcal{L}(Y,X)$ that satisfies

$$\langle\mathcal{P}f, g\rangle_Y = \langle f, \mathcal{P}^\star g\rangle_X \text{ for all } f \in X, g \in Y.$$

  The operator $\mathcal{P}^\star$ is called the ***adjoint operator*** of $\mathcal{P}$.

- The operator $\mathcal{P} \in \mathcal{L}(X,Y)$ is known as ***self-adjoint*** if $\mathcal{P} = \mathcal{P}^\star$.

- A self-adjoint operator $\mathcal{P} \in \mathcal{L}(X)$ is known as a ***positive operator***, denoted by $\mathcal{P} > 0$, if there exists a scalar $\epsilon > 0$ such that $\langle \mathcal{P}f, f \rangle_X \geq \epsilon \langle f, f \rangle_X$, for all $f \in X$.

  Similarly, a self-adjoint operator $\mathcal{P} \in \mathcal{L}(X)$ is known as a ***positive semidefinite operator***, denoted by $\mathcal{P} \geq 0$, if $\langle \mathcal{P}f, f \rangle_X \geq 0$, for all $f \in X$.

- For any two self-adjoint operators $\mathcal{P}, \mathcal{R} \in \mathcal{L}(X)$, by $P > R$ we mean that $P - R$ is a positive operator.

  Similarly, by $P \geq R$ we mean that $P - R$ is a positive semidefinite operator.

- The ***identity operator*** is denoted by $\mathcal{I}$.

- A linear operator $\mathcal{T} : \mathcal{D} \subset X \to Y$ is said to be ***closed*** if whenever

$$x_n \in \mathcal{D}, \quad n \in \mathbb{N} \quad \text{and} \quad \lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} \mathcal{T}x_n = \mathcal{T}x.$$

Vector Spaces and Real Algebra

- The set of ***non-negative real numbers*** is denoted by $\mathbb{R}^+$.

- The set of ***real matrices of dimension*** $\mathbf{m} \times \mathbf{n}$ is denoted by $\mathbb{R}^{m \times n}$.

- The set of ***symmetric matrices of dimension*** $\mathbf{n} \times \mathbf{n}$ is denoted by $\mathbb{S}^n$.

- A symmetric matrix $A \in S^n$ is a ***positive definite matrix***, denoted by $A > 0$, if there exists a scalar $\epsilon > 0$ such that $x^T A x \geq \epsilon x^T x$, for all $x \in \mathbb{R}^n$.

  Similarly, a symmetric matrix $A \in S^n$ is a ***positive semidefinite matrix***, denoted by $A \geq 0$, if $x^T A x \geq 0$, for all $x \in \mathbb{R}^n$.

- For any two symmetric matrices $A, B \in \mathbb{S}^n$, by $A > B$ we mean that $A - B$ is a positive definite matrix.

  Similarly, by $A \geq B$ we mean that $A - B$ is a positive semidefinite matrix.

- The identity matrix of dimension $n \times n$ is denoted by I_n.
- We denote by $Z_d(x)$ the ***vector of monomials up to degree*** $\mathbf{d}$.
- We denote by $Z_{n,d}(x)$ the ***Kronecker product*** $\mathbf{I_n} \otimes \mathbf{Z_d}(\mathbf{x})$.

CHAPTER 3

CONVEX OPTIMIZATION, SEMI-DEFINITE PROGRAMMING AND SUM-OF-SQUARES POLYNOMIALS

Given the functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{0, \cdots, m\}$ and $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in \{1, \cdots, p\}$, a *constrained optimization problem* can be stated as

$$\begin{aligned} \text{Minimize}_{x \in \mathbb{R}^n}: \quad & f_0(x) \\ \text{subject to}: \quad & f_i(x) \leq 0, \quad i \in \{1, \cdots, m\}, \\ & h_i(x) = 0, \quad i \in \{1, \cdots, p\}. \end{aligned} \tag{3.1}$$

The function $f_0(x)$ is the *cost function* or the *objective function*. The inequalities $f_i(x) \leq 0$ are called *inequality constraints* and the functions $f_i(x)$ are called the *inequality constraint functions*. Similarly, $h_i(x) = 0$ are the *equality constraints* and $h_i(x)$ are the *equality constraint functions*. The *optimal value* $p^\star$ of the Problem (3.1) is given as

$$p^\star = \inf\{f_0(x) : f_i(x) \leq 0, \quad i = 1, \cdots, m, \quad h_i(x) = 0, \quad i = 1, \cdots, p\}$$

and $x^\star$ for which $f_0(x^\star) = p^\star$ is the *optimal point*.

For a point $\tilde{x}$ to be an optimal point of a differentiable function $f(x)$, the necessary condition is that $[\nabla_x f(x)]_{x=\tilde{x}} = 0$, where ∇_x denotes the gradient with respect to x. The Karush-Kuhn-Tucker (KKT) conditions generalize this necessary condition for constrained optimization problems. The KKT conditions can be stated as follows [49, 50]: for the optimization Problem (3.1), with differentiable f_i and g_i, a point $x^\star \in \mathbb{R}^n$ is optimal ($f(x^\star) = p^\star$) only if there exist scalars $\lambda_i^\star$ and $\nu_i^\star$, known as *Lagrange multipliers*, such that

$$1) \quad f_i(x^\star) \leq 0, \quad i \in \{1, \cdots, m\}, \quad h_i(x^\star) = 0, \quad i \in \{1, \cdots, p\}. \tag{3.2}$$

$$2) \quad \lambda_i^\star \geq 0, \quad i \in \{1, \cdots, m\}. \tag{3.3}$$

$$3) \quad \lambda_i^\star f_i(x^\star) = 0, \quad i \in \{1, \cdots, m\}. \tag{3.4}$$

$$4) \quad \left[\nabla_x f_0(x) + \sum_{i=1}^{m} \lambda_i^\star \nabla_x f_i(x) + \sum_{i=1}^{p} \nu_i^\star \nabla_x h_i(x) \right]_{x=x^\star} = 0. \tag{3.5}$$

The solution to the equations yielded by the KKT conditions are known as *KKT points*. The KKT points are the candidate optimal points for the opimization Problem (3.1). Equations (3.2)-(3.5) can be solved numerically, although for a few cases, they can be solved analytically as well.

For a few types of optimization problems, the KKT conditions are necessary and sufficient. For example, under certain conditions, KKT conditions are necessary and sufficient for convex optimization problems. We begin by defining convex functions. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y),$$

for all $x, y \in \mathbb{R}^n$ and all $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, $\alpha \geq 0$, $\beta \geq 0$. A *convex optimization problem* can be stated as

$$\begin{aligned} \text{Minimize}_{x \in \mathbb{R}^n} : \quad & f_0(x) \\ \text{subject to} : \quad & f_i(x) \leq 0, \quad i \in \{1, \cdots, m\}, \\ & Ax = b, \quad A \in \mathbb{R}^{p \times n}, \quad b \in \mathbb{R}^p, \end{aligned} \tag{3.6}$$

where the functions f_i, $i \in \{0, \cdots, m\}$ are convex. Thus, a convex optimization problem has a convex cost function, convex inequality constraint functions and affine equality constraint functions.

Let Problem (3.6) be *strictly feasible*, i.e., there exists a point $\tilde{x} \in \mathbb{R}^n$ such that

$$f_i(\tilde{x}) < 0, \quad i \in \{1, \cdots, m\}, \quad A\tilde{x} = b. \tag{3.7}$$

Then, a point $x^\star \in \mathbb{R}^n$ is the optimal point if and only if there exist Lagrange multipliers $\lambda_i^\star$ and $\nu_i^\star$ satisfying the KKT conditions [43]. Thus, for strictly feasible convex optimization problems, the KKT conditions are necessary and sufficient for optimality.

To solve convex optimization problems, descent algorithms may be used. For the convex optimization Problem (3.6), descent algorithms produce a sequence $x^{(k)}$ satisfying $f_0(x^{(k)}) \geq f_0(x^{(k+1)}) \geq f(x^{(k+2)}) \geq \cdots$, where each element of the sequence satisfies the constraints. Given a feasible starting point $x^{(0)}$, the descent sequence is defined recursively as

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)},$$

where $t^{(k)} \geq 0$. Here $\Delta x^{(k)}$ is defined as the *search direction* and the scalar $t^{(k)}$ is the *step length.* A *valid search direction* $\Delta x^{(k)}$ is one such that for $x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}$, $f_0(x^{(k+1)}) \leq f_0(x^{(k)})$. For equality constrained optimization, Newton's method may be used. The Newton's method, at each iterate, calculates the valid descent direction by minimizing the quadratic approximation of the cost function subject to the equality constraints. Calculation of this minimizer is equivalent to solving the necessary KKT conditions, which for equality constrained optimization problems, is a system of linear equations. A detailed discussion on Newton's method can be found in [43].

To solve the constrained optimization Problem (3.6), the inequality constraints are incorporated into the cost function using a *barrier function.* Problem (3.6) can be written as

$$\begin{aligned} \text{Minimize}_{x \in \mathbb{R}^n} : &\quad f_0(x) - \sum_{i=1}^{m} \left(\frac{1}{h}\right) \log(-f_i(x)) \\ \text{subject to} : &\quad Ax = b, \quad A \in \mathbb{R}^{p \times n}, \quad b \in \mathbb{R}^p, \end{aligned} \tag{3.8}$$

where the function $\phi(u) = -\left(\frac{1}{h}\right) \log(-u)$, for some $h > 0$, is the *logarithmic barrier function.* Note that this approximate problem is convex due to the convexity of the logarithmic barrier functions. The Newton's method may now be applied to obtain the optimal point, denoted by $x^\star(h)$, for this problem. The interesting property of

$x^\star(h)$ is that

$$f_0(x^\star(h)) - p^\star \leq \frac{m}{h},$$

where $p^\star$ is the optimal value of the original Problem (3.6). Thus, as $h \to \infty$, $f(x^\star(h)) \to p^\star$. This fact is exploited by the *barrier method* and can be summarized as:

Given a feasible starting point $x^{(0)} \in \mathbb{R}^n$, $h > 0$, $\mu > 1$ and tolerance $\epsilon > 0$

repeat

1. Formulate Problem (3.6) as Problem (3.8).
2. Apply Newton's method for equality constrained convex optimization problems to Problem (3.8) to obtain $x^\star(h)$.
3. Update: $h = \mu h$ and $x^{(0)} = x^\star(h)$.

until The stopping criteria $\|\nabla f_0(x)\|_2 \leq \epsilon$ is reached.

The stopping criteria chosen is the simplest one because $\nabla f_0(x^\star) = 0$.

3.1 Semi-Definite Programming

We use Lyapunov functions parametrized by sum-of-squares polynomials for the analysis and control of parabolic PDEs. The search for such Lyapunov functions can be represented as *Semi-Definite Programming* (SDP) problems.

An SDP problem is an optimization problem of the form

$$\begin{aligned} \text{Minimize}_{x\in\mathbb{R}^n}: \quad & c^T x \\ \text{subject to}: \quad & F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \leq 0 \quad \text{and} \\ & Ax = b, \end{aligned} \tag{3.9}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, $A \in \mathbb{R}^{k\times n}$ and symmetric matrices $F_i \in \mathbb{S}^m$ are given. Since the cost function is linear and the the constraints are affine, an SDP problem is a convex optimization problem. This allows SDP problems to be solved efficiently, for example, using interior point methods. A survey of the theory and applications of SDP problems can be found in [51].

Usually, SDP problems are used to solve the feasibility problem: does there exist an $x \in \mathbb{R}^n$ such that $F(x) \leq 0$? The inequality $F(x) \leq 0$ is linear in the search variables. Thus, the feasibility problem is known as a *Linear Matrix Inequality* (LMI). Any number of given LMIs can be cast as a single LMI. For example, LMIs $F(x) \leq 0$ and $G(x) \leq 0$ can be rewritten as

$$\begin{bmatrix} F(x) & 0 \\ 0 & G(x) \end{bmatrix} = \begin{bmatrix} F_0 & 0 \\ 0 & G_0 \end{bmatrix} + \sum_{i=1}^{n} x_i \begin{bmatrix} F_i & 0 \\ 0 & G_i \end{bmatrix} \leq 0.$$

Another example of LMIs arise in finite-dimensional control theory. The linear time invariant system

$$\dot{x}(t) = Ax(t), \quad A \in \mathbb{R}^{n \times n}$$

is stable if and only if there exists a symmetric matrix $X \in \mathbb{S}^n$ such that [52, Corollary 4.3]

$$X > 0 \quad \text{and} \quad A^T X + XA < 0. \tag{3.10}$$

The search for the positive definite X can be cast as an LMI. Let

$$X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}.$$

Then

$$X = x_1 e_{11} + x_2(e_{12} + e_{21}) + x_3 e_{22},$$

where $e_{ij} \in \mathbb{S}^2$ are matrices with $e(i,j) = 1$ and zeros everywhere else. Thus, the conditions in Equation (3.10) can be cast as the following LMI

$$F(x) = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sum_{i=1}^{3} x_i F_i \leq 0,$$

where

$$F_1 = \begin{bmatrix} -e_{11} & 0 \\ 0 & A^T e_{11} + e_{11} A \end{bmatrix}, \quad F_2 = \begin{bmatrix} -(e_{12} + e_{21}) & 0 \\ 0 & A^T(e_{12} + e_{21}) + (e_{12} + e_{21})A \end{bmatrix},$$

$$F_3 = \begin{bmatrix} -e_{22} & 0 \\ 0 & A^T e_{22} + e_{22} A \end{bmatrix}$$

and $\epsilon > 0$.

Since SDP problems are convex, they can be solved efficiently using convex optimization algorithms. For example, interior point methods are widely used for solving SDPs [53], [54], [44].

3.2 Sum-of-Squares Polynomials

Sum-of-Squares (SOS) is an approach to the optimization of positive polynomial variables. A typical formalism for the polynomial optimization problem is given by

$$\max_x \; c^T x, \quad \text{subject to} \qquad \sum_{i=1}^{m} x_i f_i(y) + f_0(y) \geq 0,$$

for all $y \in \mathbb{R}^n$, where the f_i are real polynomial functions. The key difficulty is that the feasibility problem of determining whether a polynomial is globally positive ($f(y) \geq 0$ for all $y \in \mathbb{R}^n$) is NP-hard [55]. This means that there are no algorithms which can determine the global positivity of polynomials in polynomial time. Thus, relaxations that are tractable for such problems are required. A particularly important such condition is that the polynomial be sum-of-squares.

Definition 3.1. *A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is* ***Sum-of-Squares*** *(SOS) if there exist polynomials $g_i : \mathbb{R}^n \to \mathbb{R}$ such that*

$$p\,(x) = \sum_i g_i^2(x).$$

We use $p \in \Sigma_s$ to denote that p is SOS.

The importance of the SOS condition lies in the fact that it can be readily enforced using semidefinite programming. This fact is attributed to the following theorem.

Theorem 3.2. *A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree $2d$ is SOS if and only if there exists a Positive Semi-Definite (PSD) matrix Q such that*

$$p(x) = Z_d^T(x) Q Z_d(x), \tag{3.11}$$

where $Z_d(x)$ is a vector of monomials up to degree d.

Proof. **If:** Since Q is PSD, there exists a matrix A such that $Q = A^\star A$, where $A^\star$ is the conjugate transpose of A. Hence, we have

$$p(x) = Z_d^T(x) A^\star A Z_d(x) = (A Z_d(x))^\star A Z_d(x).$$

It can be easily seen that $AZ_d(x) = G(x)$ is a vector of polynomials. Thus

$$p(x) = G(x)^\star G(x) \in \Sigma_s.$$

Only if: Since $p \in \Sigma_s$, there exist polynomials $g_i : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$p(x) = \sum_i g_i^2(x).$$

Let $G^T(x) = [g_1(x), \cdots, g_i(x)]$. Then,

$$p(x) = G^T(x) G(x).$$

Now, $g_i(x) = a_i^T Z_d(x)$, where a_i is the vector containing the coefficients of the polynomial $g_i(x)$. Thus

$$G(x) = \begin{bmatrix} a_1^T \\ \vdots \\ a_i^T \end{bmatrix} Z_d(x) = A^T Z_d(x).$$

Hence

$$p(x) = G^T(x)G(x) = Z_d^T(x)AA^T Z_d(x) = Z_d^T(x)QZ_d(x).$$

The observation that $Q = AA^T$, and hence is a PSD matrix, completes the proof. $\square$

A proof similar to the one we present can be found in [46].

As a simple example consider the polynomial $p(x) = a^2 + b^2x^2 + 2abx$, for arbitrary scalars a and b. Then, $p \in \Sigma_s$ since $p(x) = (a + bx)^2$. Additionally, for $Z_1^T(x) = [1 \quad x]$, we have

$$p(x) = Z_1^T(x) \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} Z_1(x) = Z_1^T(x)QZ_1(x),$$

where Q is PSD for any $a, b \in \mathbb{R}$.

Theorem 3.2 establishes the link between SOS polynomials and PSD matrices. In this way optimization of positive polynomials can be converted to SDP. The SDP approach to polynomial positivity was described in the thesis work of [46] and also in [56]. See also [57] and [58] for contemporaneous work. MATLAB toolboxes for manipulation of SOS variables have been developed and can be found in [59] and [60].

Note that the condition that a polynomial is globally positive if it is SOS is conservative. This is because not all globally positive polynomials are SOS. A detailed discussion on this topic can be found in [46]. A well known example of a positive polynomial which is not SOS is the Motzkin polynomial $x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3^2$. Proof of the Motzkin polynomial's global positivity can be found in literature. It was demonstrated in [46] that there exists no PSD matrix satisfying Equation (3.11) for the Motzkin polynomial.

SOS polynomials can be used for the stability analysis of non-linear systems of the type

$$\dot{x}(t) = f(x(t)), \tag{3.12}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial satisfying $f(0) = 0$. The condition for the global asymptotic stability of $x = 0$ is that there exist a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, for some $\epsilon > 0$, satisfying

$$\begin{aligned} &V(x(t)) - \epsilon x(t)^T x(t) \geq 0, \\ &\nabla V(x(t))^T f(x) - \epsilon x(t)^T x(t) \leq 0. \end{aligned}$$

As previously stated, showing the global positivity of polynomials is intractable. However, we can use SOS polynomials to relax the conditions to [46]:

$$\begin{aligned} &V(x(t)) - \epsilon x(t)^T x(t) \in \Sigma_s \\ &- \nabla V(x(t))^T f(x) - \epsilon x(t)^T x(t) \in \Sigma_s, \end{aligned}$$

for some $\epsilon > 0$. This membership can be now tested in polynomial time using, for example, SOSTOOLS [59].

3.2.1 Postivstellensatz.

A positivstellensatz is a theorem from real algebraic geometry which provides a means to verify polynomial positivity over semialgebraic sets.

Definition 3.3. *A* ***semialgebraic set*** *is a set of the form*

$$\mathcal{S} = \{x \in \mathbb{R}^n : \quad g_i(x) \geq 0, \quad i \in \{1, \cdots, m\}, \quad h_i(x) = 0, \quad i \in \{1, \cdots, p\}\},$$

where each g_i and h_i is a real valued polynomial.

The closed unit disc in $\mathbb{R}^2$ is a straightforward example of a semialgebraic set defined as

$$\mathcal{S} = \{x \in \mathbb{R}^2 : \quad -x_1^2 - x_2^2 + 1 \geq 0\}.$$

We are asking the following feasibility question: Given a semialgebraic set $\mathcal{S}$, is there a polynomial $f(x)$ such that $f(x) \geq 0$, for all $x \in \mathcal{S}$? Of course, if the polynomials f and g_i are convex, and h_i are affine, then we have a convex feasibility problem.

Theorem 3.4 (Stengle's positivstellensatz, [61])**.** *Given the polynomials $g_i(x)$, $i \in \{1, \cdots, m\}$, let*

$$\mathcal{S} = \{x \in \mathbb{R}^n : \quad g_i(x) \geq 0, \quad i \in \{1, \cdots, m\}\}.$$

Then, $\mathcal{S} = \emptyset$ if and only if there exist $s_i, s_{ij}, s_{ijk}, \cdots, s_{ijk\cdots m} \in \Sigma_s$ such that

$$\begin{aligned} -1 =& s_0(x) + \sum_i s_i(x) g_i(x) + \sum_{i \neq j} s_{ij}(x) g_i(x) g_j(x) \\ &+ \sum_{i \neq j \neq k} s_{ijk}(x) g_i(x) g_j(x) g_k(x) + \cdots + s_{ijk\cdots m}(x) g_i(x) g_j(x) g_k(x) \cdots g_m(x). \end{aligned}$$

The following corollary expresses the conditions of polynomial positivity on a semialgebraic set.

Corollary 3.5. *Given the polynomials $f(x)$ and $g_i(x)$, $i \in \{1, \cdots, m\}$, $f(x) > 0$, for all $x \in \{x \in \mathbb{R}^n : \quad g_i(x) \geq 0\}$, if and only if there exist*

$$p_0, s_i, p_{ij}, s_{ij}, p_{ijk}, s_{ijk}, \cdots, p_{ijk\cdots m}, s_{ijk\cdots m} \in \Sigma_s$$

such that

$$\begin{aligned} f(x) \Bigg(& p_0 + \sum_{i \neq j} p_{ij}(x) g_i(x) g_j(x) + \sum_{i \neq j \neq k} p_{ijk}(x) g_i(x) g_j(x) g_k(x) \\ & + \cdots + p_{ijk\cdots m}(x) g_i(x) g_j(x) g_k(x) \cdots g_m(x) \Bigg) \end{aligned}$$

$$= 1 + s_0(x) + \sum_i s_i(x)g_i(x) + \sum_{i \neq j} s_{ij}(x)g_i(x)g_j(x)$$
$$+ \sum_{i \neq j \neq k} s_{ijk}(x)g_i(x)g_j(x)g_k(x) + \cdots + s_{ijk\cdots m}(x)g_i(x)g_j(x)g_k(x)\cdots g_m(x).$$

Proof. The condition that $f(x) > 0$, for all $x \in \{x \in \mathbb{R}^n : \quad g_i(x) \geq 0\}$, is equivalent to the emptiness of the set

$$\mathcal{S} = \{x \in \mathbb{R}^n : \quad -f(x) \geq 0, \quad g_i(x) \geq 0, \quad i \in \{1, \cdots, m\}\}.$$

Thus, the result is obtained by applying Theorem 3.4 to the semialgebraic set $\mathcal{S}$. $\square$

This corollary can be used to test polynomial positivity on a semialgebraic set. However, although the search of the SOS multipliers can be cast as an LMI, the equality constraint is no longer affine in the search variables f, s and p. In fact, it is bilinear. Hence, this check cannot be performed using semidefinite programming.

When the semialgebraic sets are compact, the following positivstellensatz conditions hold.

Theorem 3.6 (Schmudgen's positivstellensatz, [62]). *Given the polynomials $f(x)$ and $g_i(x)$, $i \in \{1, \cdots, m\}$, let*

$$\mathcal{S} = \{x \in \mathbb{R}^n : \quad g_i(x) \geq 0, \quad i \in \{1, \cdots, m\}\}$$

be compact. If $f(x) > 0$, for all $x \in \mathcal{S}$, then there exist $s_i, s_{ij}, s_{ijk}, \cdots, s_{ijk\cdots m} \in \Sigma_s$ such that

$$f(x) =1 + s_0(x) + \sum_i s_i(x)g_i(x) + \sum_{i \neq j} s_{ij}(x)g_i(x)g_j(x)$$
$$+ \sum_{i \neq j \neq k} s_{ijk}(x)g_i(x)g_j(x)g_k(x) + \cdots + s_{ijk\cdots m}(x)g_i(x)g_j(x)g_k(x)\cdots g_m(x).$$

Now, the equality constraint is affine in f and s. Thus, Schmudgen's positivstellensatz can be tested using semidefinite programming.

Definition 3.7. *Given the polynomials $g_i(x)$, $i \in \{1, \cdots, m\}$, the set*

$$\mathcal{M}(g_i) = \{p_0(x) + \sum_{i=1}^{m} p_i(x) g_i(x), \quad p_0, p_i \in \Sigma_s\}$$

is called the ***quadratic module generated by*** $\mathbf{g_i}$.

Theorem 3.8 (Putinar's positivstellensatz, [63])**.** *Given the polynomials $g_i(x)$, $i \in \{1, \cdots, m\}$, suppose there exists a polynomial $h \in \mathcal{M}(g_i)$ such that*

$$\{x \in \mathbb{R}^n : \quad h(x) \geq 0\} \tag{3.13}$$

is a compact set. Then, if $f(x) \geq 0$, for all $x \in \mathcal{S}$, where

$$\mathcal{S} = \{x \in \mathbb{R}^n : \quad g_i(x) \geq 0, \quad i \in \{1, \cdots, m\}\},$$

there exist $s_0, s_i \in \Sigma_s$ such that

$$f(x) = s_0(x) + \sum_i s_i(x) g_i(x).$$

Equivalent conditions, which are also semidefinite programming verifiable, for the one in Equation (3.13) can be found in [64]. Similar to Theorem 3.6, the conditions of Theorem 3.8 can be checked using semidefinite programming. In terms of computational complexity, it can be seen that Putinar's positivstellensatz requires a much smaller number of SOS multipliers compared to Schmudgen's and Stengle's positivstellensatz.

A summary of positivstellensatz results can be found in [65].

We can use positivstellensatz results for the local stability analysis of the system given by

$$\dot{x}(t) = f(x(t)),$$

with polynomial f, on the semialgebraic set given by

$$\mathcal{S} = \{x \in \mathbb{R}^n : \quad g_i(x) \geq 0, \quad i \in \{1, \cdots, m\}\}.$$

We can now search for a polynomial Lyapunov function $V(x(t))$, scalar $\epsilon > 0$ and SOS polynomials s_0, p_0, s_i and p_i such that

$$\begin{aligned} V(x(t)) - \epsilon x(t)^T x(t) &= s_0(x) + \sum_i s_i(x) g_i(x), \\ -\nabla V(x(t))^T f(x) - \epsilon x(t)^T x(t) &= p_0(x) + \sum_i p_i(x) g_i(x). \end{aligned}$$

CHAPTER 4

POLOIDAL MAGNETIC FLUX MODEL

The critical physical quantity in a tokamak is the magnetic field which is a combination of the toroidal magnetic field B_T and the poloidal magnetic field B_P. The toroidal magnetic field B_T is controlled by powerful external current carrying coils. Whereas, the poloidal magnetic field is generated by the plasma current I_p. Consequently, the polidal magnetic field is an order of magnitude smaller than the toroidal magnetic field [6]. The coupling with the plasma current makes the poloidal magnetic field vulnerable to changes in the plasma. Additionally, regulating a suitable plasma current profile by regulating the poloidal magnetic flux has been demonstrated as an important condition for improved plasma confinement and steady state operation [66].

Let $\psi(R, Z)$ denote the flux of the magnetic field passing through a disc centered on the toroidal axis at a height Z with the surface area πR^2 as depicted in Figure 4.1. The simplified dynamics of the poloidal flux $\psi(\rho, t)$ are given by [67]:

$$\psi_t(\rho, t) = \frac{\eta_{\|} C_2}{\mu_0 C_3} \psi_{\rho\rho} + \frac{\eta_{\|} \rho}{\mu_0 C_3^2} \frac{\partial}{\partial \rho} \left(\frac{C_2 C_3}{\rho} \right) \psi_\rho + \frac{\eta_{\|} V_\rho B_{\phi_0}}{F C_3} j_{ni}, \tag{4.1}$$

where the spatial variable $\rho := \left(\frac{\phi}{\pi B_{\phi_0}} \right)^{\frac{1}{2}}$ (ϕ being the toroidal magnetic flux and B_{ϕ_0} the toroidal magnetic flux at the center of the vacuum vessel of the tokamak) is the radius indexing the magnetic surfaces, $\eta_{\|}$ is the parallel resistivity of the plasma, j_{ni} is the non-inductively deposited current density, μ_0 is the permeability of free space, F is the diamagnetic function, C_2 and C_3 are geometric coefficients, V_ρ is the spatial derivative of the plasma volume and B_{ϕ_0} is the toroidal magnetic field at the geometric center of the plasma. The various variable definitions are provided in Table 4.1.

Neglecting the diamagnetic effect applying cylindrical approximation of the plasma geometry ($\rho << R_0$, where R_0 is the major plasma radius) the coefficients in

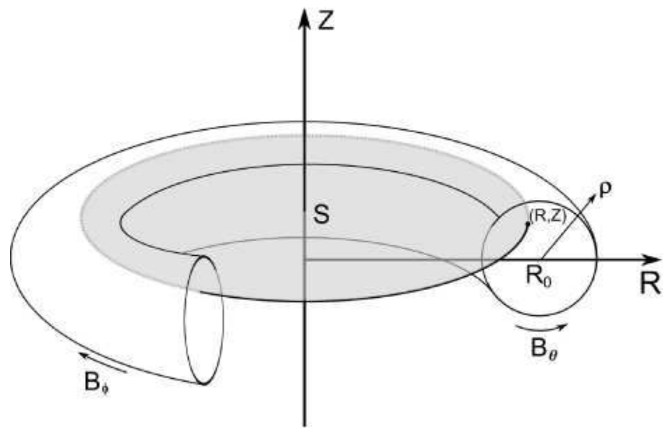


Figure 4.1. Coordinates (R, Z) and surface S used to define the poloidal magnetic flux $\psi(R, Z)$ [68].

Equation (4.1) simplify as follows:

$$F \approx R_0 B_{\phi_0}, \quad C_2 = C_3 = 4\pi^2 \frac{\rho}{R_0}, \quad V_\rho = 4\pi^2 \rho R_0.$$

Defining a normalized spatial variable $x = \rho/a$, where a is the radius of the last closed magnetic surface and is assumed to be constant, the simplified model is obtained as in [47]:

$$\psi_t(x,t) = \frac{\eta_\parallel(x,t)}{\mu_0 a^2}\left(\psi_{xx} + \frac{1}{x}\psi_x\right) + \eta_\parallel(x,t) R_0 j_{ni}(x,t) \tag{4.2}$$

with boundary conditions

$$\psi_x(0,t) = 0 \quad \text{and} \quad \psi_x(1,t) = -\frac{R_0 \mu_0 I_p(t)}{2\pi}. \tag{4.3}$$

The diffusion coefficient in Equation (4.2) is the plasma parallel resistivity $\eta_\parallel$. The plasma resistivity introduces a coupling between the poloidal magnetic flux ψ, the electron temperature profile T_e and the electron density profile n_e as follows. The expression for the resistivity is computed using the results in [69] by using the expressions for the electron thermal speed α_e and the electron collision time τ_e, given in [6], as

$$\alpha_e(x,t) = \sqrt{\frac{eT_e}{m_e}} \quad \text{and} \quad \tau_e(x,t) = \frac{12\pi^{3/2} m_e^{1/2} \epsilon_0^2}{e^{5/2}\sqrt{2}} \frac{T_e^{3/2}}{n_e \log \Lambda},$$

where $e = 1.6022 \times 10^{-19} C$ is the electron charge, $m_e = 9.1096 \times 10^{31} kg$ is the electron mass and $\epsilon_0 = 8.854 \times 10^{-12} F m^{-1}$ is the permittivity of free space. Additionally, $\Lambda(x,t) = 31.318 + \log(T_e/\sqrt{n_e})$. Using these two expressions, the parallel conductivity can be calculated as [47]:

$$\sigma_{\|}(x,t) = \sigma_0 \Lambda_E \left(1 - \frac{f_t}{1+\xi\nu}\right)\left(1 - \frac{C_R f_t}{1+\xi\nu}\right),$$

where

$$\sigma_0(x,t) = \frac{n_e e^2 \tau_e}{m_e}, \quad \Lambda_E(\bar{Z}) = \frac{3.40}{\bar{Z}}\left(\frac{1.13+\bar{Z}}{2.67+\bar{Z}}\right), \qquad \nu(x,t) = \frac{R_0 B_{\phi_0} a^2 x}{(x\epsilon)^{3/2}\alpha_e \tau_e \psi_x},$$
$$f_t(x) = 1 - (1-x\epsilon)^2(1-(x\epsilon)^2)^{-1/2}(1+1.46\sqrt{x\epsilon})^{-1},$$
$$\xi(\bar{Z}) = 0.58 + 0.20\bar{Z}, \quad C_R(\bar{Z})\frac{0.56}{\bar{Z}}\left(\frac{3-\bar{Z}}{3+\bar{Z}}\right),$$

and $\bar{Z}$ is the effective value of the plasma charge. With the expression for the parallel conductivity $\sigma_{\|}$, the expression for the parallel resistivity $\eta_{\|}$ and be calculated as

$$\eta_{\|}(x,t) = \frac{1}{\sigma_{\|}(x,t)}.$$

The plasma current I_p is generated by the electromagnetic induction by the central ohmic coil. In addition, plasma current is also generated by non-inductive sources. The current generated by non-inductive means is known as the *current drive* (j_{ni} in Equation (4.2)). The non-inductive current has two main components: the internally generated *bootstrap current density* j_{bs} and the *external non-inductive current density* j_{eni}. We will discuss these current drive sources briefly.

The magnetic field strength in a tokamak, due to the vessel being toroidal, is proportional to $1/R$ as given by Ampere's law. Thus, the magnetic field strength is stronger on the inside of the tokamak vessel as compared to the outside. Since the ions and electrons follow the helical magnetic field lines around the toroid, they transition from the weak magneic field side to the strong side and vice-versa. In the absence of enough particle velocity parallel to the magnetic lines, a particle undergoes a magnetic mirror reflection [6]. Such particles remain trapped in the weak field side of the tokamak and thus, instead of going around in the poloidal plane, are forced to orbit the weaker magnetic side of the poloidal plane in what is known as *banana orbits*. The trapping of a few particles leads to collision between the trapped and free particles owing to their different orbits. These collisions lead to a momentum transfer between the trapped and free particles generating a current density which is known as the bootstrap current density [30], [70].

The model for the bootstrap current density is given in [71] as

$$j_{bs}(x,t) = \frac{p_e R_0}{\psi_x}\left[A_1\left[\frac{1}{p_e}\frac{\partial p_e}{\partial x} + \frac{p_i}{p_e}\left(\frac{1}{p_i}\frac{\partial p_i}{\partial x} - \alpha_i\frac{1}{T_i}\frac{\partial T_i}{\partial x}\right)\right] - A_2\frac{1}{T_e}\frac{\partial T_e}{\partial x}\right],$$

where p_e and p_i are the electron and ion pressure profiles respectively, T_e and T_i are the electron and ion temperature profiles respectively, α_i is the ion thermal speed and the A_1 and A_2 are functions of the ratio of trapped to free particles. We can use the expressions $p_e = en_eT_e$ and $p_i = en_iT_i$ to express the bootstrap current density in terms of temperature and density profiles as

$$j_{bs}(x,t) = \frac{eR_0}{\psi_x}\left((A_1 - A_2)n_e\frac{\partial T_e}{\partial x} + A_1T_e\frac{\partial n_e}{\partial x} + A_1(1-\alpha_i)n_i\frac{\partial T_i}{\partial x} + A_1T_i\frac{\partial n_i}{\partial x}\right). \tag{4.4}$$

The fraction of the total current due to bootstrap current can also be estimated using the empirical expression derived in [72].

The externally generated current density j_{eni} has two components: the *Lower Hybrid Current Density* (LHCD) denoted by j_{lh}, and the *Electron Cyclotron Current Density* (ECCD) denoted by j_{ec}. The actuators for these current density deposits are Radio Frequency (RF) antennas. The ECCD actuator is tuned to the electron cyclotron resonant frequency and the LHCD actuator is tuned to a frequency which lies between the electron and ion cyclotron resonant frequencies [6]. We only consider the LHCD current density deposit j_{lh}, although, the work presented can easily be extended to include ECCD as well.

The LHCD input $j_{lh}(x,t)$ is a function of the control actuator parameters $N_{\parallel}$, the hybrid wave parallel refractive index, and P_{lh}, the lower hybrid antenna power. The development of an expression for the LHCD input is particularly difficult since the LHCD deposit depends on the operating conditions [73]. One way of estimating the LHCD deposit profile is to use X-ray measurements of electrons to develop an empirical expression [74]. Using the X-ray measurements from the *Tore Supra* tokamak, an empirical model of the LHCD current density deposition was developed in [47]. This model uses a Gaussian deposition pattern with control authority over certain scaling parameters. The width $w(t)$ and center $\mu(t)$ of the deposit can be estimated as [47]:

$$\begin{aligned} w(t) =& 0.53 B_{\phi_o}^{-0.24} I_p^{0.57} \bar{n}^{-0.08} P_{LH}^{0.13} N_{\parallel}^{0.39} \\ \mu(t) =& 0.20 B_{\phi_o}^{-0.39} I_p^{0.71} \bar{n}^{-0.02} P_{LH}^{0.13} N_{\parallel}^{1.20}. \end{aligned}$$

The total current deposit can be established using the empirical laws presented in [75] as

$$I_{LH}(t) = \frac{\eta_{LH} P_{LH}}{\bar{n} R_0},$$

where $\eta_{LH}(t) = 1.18 D_n^{0.55} I_p^{0.43} \bar{Z}^{-0.24}$ and $D_n(t) \approx 2.03 - 0.63 N_{\parallel}$. The expression for j_{LH} can now be given as

$$j_{LH}(x,t) = v_{LH}(t) e^{-(\mu(t)-x)^2/2\sigma_{LH}(t)},$$

where

$$v_{LH}(t) = I_{LH}(t) \left(2\pi a^2 \int_0^1 x e^{-(\mu(t)-x)^2/2\sigma_{LH}(t)} dx \right)^{-1} \quad \text{and} \quad \sigma_{LH}(t) = \frac{(\mu(t) - w(t))^2}{2 \log 2}.$$

The *safety factor profile*, or the *q-profile*, is the magnetic field line pitch [6]. The q-profile is a common heuristic for setting operating conditions that avoid Magneto-Hydro-Dynamic (MHD) instabilities [76]. The q-profile is defined as the ratio of the toroidal and poloidal magnetic flux gradients. The safety factor profile is defined in terms of the gradient of the poloidal magnetic flux ψ_x as [47]:

$$q(x,t) = \frac{\phi_x}{\psi_x} = -\frac{B_{\phi_0} a^2 x}{\psi_x}, \tag{4.5}$$

where B_{ϕ_0} is the toroidal magnetic flux at the plasma center. Thus, to control the q-factor profile, gradient of the poloidal magnetic flux $\psi_x(x,t)$ may be controlled. The model for the evolution of $Z = \psi_x$ can be obtained by differentiating Equation (4.2) in x to get

$$\frac{\partial Z}{\partial t}(x,t) = \frac{1}{\mu_0 a^2} \frac{\partial}{\partial x}\left(\frac{\eta_\|(x,t)}{x}\frac{\partial}{\partial x}\left(xZ(x,t)\right)\right) + R_0 \frac{\partial}{\partial x}\left(\eta_\|(x,t) j_{ni}(x,t)\right) \tag{4.6}$$

with boundary conditions

$$Z(0,t) = 0 \text{ and } Z(1,t) = -R_0 \mu_0 I_p(t)/2\pi. \tag{4.7}$$

Note that the control of $Z = \psi_x$ also facilitates in the control of the bootstrap current density since, from Equation (4.4), $j_{bs} \propto 1/\psi_x$.

In Chapters 8 and 9 we will devise methodologies to control the gradient of the poloidal magnetic flux. We control ψ_x to regulate the safety factor profile q and maximize the bootstrap current density j_{bs}.

Table 4.1. Tokamak plasma variable definitions.

Variables	*Description*	*Units*
ψ	Poloidal magnetic flux profile	Tm^2
ϕ	Toroidal magnetic flux profile	Tm^2
q	Safety factor profile	
R_0	Location of magnetic center	m
B_{ϕ_0}	Toroidal magnetic field at the plasma center	T
ρ	Equivalent radius of the magnetic surfaces	m
a	Location of the last closed magnetic surface	m
x	Normalized spatial variable $x \doteq \rho/a$	
V	Plasma volume	m^3
F	Diamagnetic function	Tm
C_2, C_3	Geometric coefficients	
$\eta_\parallel$	Parallel resistivity	Ωm
μ_0	Permeability of free space: $4\pi \times 10^{-7}$	Hm^{-1}
j_{ni}	Non-inductive effective current density	Am^{-2}
j_{lh}	LHCD current density	Am^{-2}
j_{bs}	Bootstrap current density	Am^{-2}
I_p	Total plasma current	A
P_{lh}	Lower hybrid antenna power	A
$N_\parallel$	Hybrid wave parallel refractive index	
m_e	Electron mass, 9.1096×10^{31}	kg
n_e	Electron density profile	m^{-3}
n_i	Electron density profile	m^{-3}
$\bar{n}$	Electron line average density	m^{-2}
T_e	Electron temperature profile	eV
T_i	Ion temperature profile	eV
τ_e	Electron collision time	s
$\bar{Z}$	Effective value of plasma charge	C
α_e	Electron thermal speed	ms^{-1}
α_i	Ion thermal speed	ms^{-1}

CHAPTER 5

STABILITY ANALYSIS OF PARABOLIC PDES

In this chapter we analyze the stability of a particular class of parabolic PDEs. The goal is to develop a methodology to check the stability and construct Lyapunov functions which act as certificates of stability. We accomplish these tasks by constructing Lyapunov functions using positive operators parametrized by sum-of-squares-polynomials.

We consider the following type of parabolic PDEs

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t), \quad x \in [0,1], \quad t \geq 0, \tag{5.1}$$

with boundary conditions of the form

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{and} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = 0. \tag{5.2}$$

The functions a, b and c are polynomial functions in x. Moreover, the function a satisfies

$$a(x) \geq \alpha > 0, \quad \text{for} \quad x \in [0,1]. \tag{5.3}$$

The scalars $\nu_i, \rho_j \in \mathbb{R}$, $i,j \in \{1,2\}$, can be chosen so that (5.2) represents Dirichlet, Neumann or Robin boundary conditions. Additionally, these scalars satisfy

$$|\nu_1| + |\nu_2| > 0 \quad \text{and} \quad |\rho_1| + |\rho_2| > 0. \tag{5.4}$$

For PDEs in the form of Equations (5.1)-(5.2), we define the first-order differential form

$$\dot{\mathbf{w}}(t) = \mathcal{A}\mathbf{w}(t), \quad \mathbf{w} \in \mathcal{D}_0 \tag{5.5}$$

where the operator $\mathcal{A} : H^2(0,1) \to L_2(0,1)$ is defined as

$$\left(\mathcal{A}y\right)(x) = a(x)y_{xx}(x) + b(x)y_x(x) + c(x)y(x), \tag{5.6}$$

and

$$\mathcal{D}_0 = \{y \in H^2(0,1) : \quad \nu_1 y(0) + \nu_2 y_x(0) = 0 \text{ and } \rho_1 y(1) + \rho_2 y_x(1) = 0\}. \tag{5.7}$$

For later use, we present the following parametrization of all possible boundary conditions.

Definition 5.1. *Given scalars ν_1, ν_2, ρ_1 and ρ_2, we define*

$$\{n_1, n_2, n_3\} = \begin{cases} \{-\frac{\nu_1}{\nu_2}, 0, 1\} & \text{if } \nu_1, \nu_2 \neq 0 \\ \{0, 1, 0\} & \text{if } \nu_1 \neq 0, \nu_2 = 0 \\ \{0, 0, 1\} & \text{if } \nu_1 = 0, \nu_2 \neq 0 \end{cases}$$

and

$$\{n_4, n_5, n_6\} = \begin{cases} \{-\frac{\rho_1}{\rho_2}, 0, 1\} & \text{if } \rho_1, \rho_2 \neq 0 \\ \{0, 1, 0\} & \text{if } \rho_1 \neq 0, \rho_2 = 0 \\ \{0, 0, 1\} & \text{if } \rho_1 = 0, \rho_2 \neq 0 \end{cases} .$$

With this definition, the boundary conditions for any $w \in \mathcal{D}_0$ can be represented as

$$w_x(0) = n_1 w(0) + n_2 w_x(0), \quad w(0) = n_3 w(0),$$
$$w_x(1) = n_4 w(1) + n_5 w_x(1), \quad w(1) = n_6 w(1).$$

5.1 Uniqueness and Existence of Solutions

We will use semigroup theory presented in Subsection A.1.1 in Appendix A to show that a classical solution of the system represented by Equation (5.5) exists. Thus, we have to show that the pair $(\mathcal{A}, \mathcal{D}_0)$ generates a C_0-semigroup. The idea is to express the operator $\mathcal{A}$ as the negative of a Sturm-Liouville operator and then use its spectral properties to show that $(\mathcal{A}, \mathcal{D}_0)$ generates a C_0-semigroup.

Definition 5.2. *[77, Chapter 8] An operator $\mathcal{S} : \mathcal{D}_0 \to L_2(0,1)$ is called a **Sturm-Liouville operator** if*

$$(\mathcal{S}y)(x) = -\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x), \quad y \in \mathcal{D}_0, \tag{5.8}$$

where p, dp/dx and q are real valued and continuous functions on $[0,1]$ and $p(x) \geq p_0 > 0$, for all $x \in [0,1]$.

Additionally, for a given $\sigma(x) > 0$, an equation of the form

$$-\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x) = \lambda\sigma(x)y(x), \tag{5.9}$$

*where $\lambda \in \mathbb{R}$, is called a **Sturm-Liouville equation**. If there exist scalars λ_n and functions ϕ_n such that*

$$-\frac{d}{dx}\left(p(x)\frac{d\phi_n(x)}{dx}\right) + q(x)\phi_n(x) = \lambda_n\sigma(x)\phi_n, \quad n \in \mathbb{N}, \tag{5.10}$$

*then, the scalars λ_n are called the **eigenvalues** of $\mathcal{S}$, and the functions ϕ_n are called the **eigenfunctions** of $\mathcal{S}$.*

The following lemma summarizes some of the spectral properties of a Sturm-Liouville operator.

Lemma 5.3. *[78] Let $\mathcal{S} : \mathcal{D}_0 \to L_2(0,1)$ be a Sturm-Liouville operator. Then, the following properties hold:*

1. *$\mathcal{S}$ is a closed operator*[1].
2. *The eigenvalues $\{\lambda_n, n \geq 0\}$ of $\mathcal{S}$ exist, are real, countable and simple.*
3. *The set of normalized eigenfunctions of $\mathcal{S}$, $\{\phi_n, n \geq 0\}$, is an orthonormal basis of $L_2(0,1)$.*

[1] Refer to Section 2.2 for the definition of a closed operator.

4. *The closure of the set* $\{\lambda_n, n \geq 0\}$ *is totally disconnected, that is, for two points* $\omega_0, \omega_1 \in \overline{\{\lambda_n, n \geq 0\}}$, $[\omega_0, \omega_1] \notin \overline{\{\lambda_n, n \geq 0\}}$.

5. *The eigenvalues* λ_n *satisfy*

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \infty \quad \textit{and} \quad \lambda_n \to \infty \textit{ as } n \to \infty.$$

Lemma 5.4. *For any initial condition* $w_0 \in \mathcal{D}_0$ *there exists a classical solution for the system represented by Equations* (5.1)-(5.2). *Additionally, for any initial condition* $w_0 \in L_2(0,1)$ *there exists a weak solution for the system represented by Equations* (5.1)-(5.2).

Proof. For the operator $\mathcal{A}$ given in (5.6), if we choose

$$p(x) = e^{\int_0^x \frac{b(\xi)}{a(\xi)} d\xi}, \quad q(x) = -c(x)\frac{p(x)}{a(x)}, \quad \sigma(x) = \frac{p(x)}{a(x)},$$

then

$$-\mathcal{A}y = \frac{1}{\sigma(x)}\mathcal{S}y, \quad y \in \mathcal{D}_0,$$

where $\mathcal{S}$ is the Sturm-Liouville operator. Therefore, using Lemma 5.3(5) and [45, Theorem 2.3.5(c)] we get that the pair $(\mathcal{A}, \mathcal{D}_0)$ is the generator of a C_0-semigroup $S(t)$ on $L_2(0,1)$.

From Theorem A.3 we obtain that for any $w_0 \in \mathcal{D}_0$, Equation (5.5), and thus (5.1)-(5.2), has a classical solution given by

$$w(t,x) = \left(S(t)w_0\right)(x). \tag{5.11}$$

□

From Corollary A.4, for any $w_0 \in L_2(0,1)$, (5.11) is the unique weak solution of (5.1)-(5.2).

5.2 Positive Operators and Semi-Separable Kernels

As stated earlier, we establish the stability of the systems under consideration by constructing Lyapunov functions parametrized by positive operators. In particular, we construct positive operators on $L_2(0,1)$ which are parametrized by Sum-of-Squares (SOS) polynomials. Since the search for SOS polynomials can be cast as a semi-definite programming as explained in Chapter 3, this parametrization allows us to construct the Lyapunov functions algorithmically.

We consider operators of the form

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \tag{5.12}$$

where $M(x) : [0,1] \to \mathbb{R}$ and $K_1(x,\xi), K_2(x,\xi) : [0,1] \times [0,1] \to \mathbb{R}$ are polynomials and $y \in L_2(0,1)$. In [79], the necessary and sufficient conditions for positivity of multiplier and integral operators of similar form using pointwise constraints on the functions M, K_1 and K_2 are given. Recently, in [80], these conditions was sharpened - See Theorem 5.5.

Theorem 5.5. *Given $d_1, d_2 \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$, let $Z_1(x) = Z_{d_1}(x)$ and $Z_2(x,\xi) = Z_{d_2}(x,\xi)$ as defined in Section 2.2. Suppose there exists a matrix U such that*

$$U = \begin{bmatrix} U_{11} - \epsilon I_0 & U_{12} & U_{13} \\ \star & U_{22} & U_{23} \\ \star & \star & U_{33} \end{bmatrix} \geq 0,$$

where I_0 is a matrix of zeros of appropriate dimensions except at the 1-by-1 element which has a value of 1, and U_{ij} are a partition of U. Let M, K_1 and K_2 be polynomials such that, for $(x,\xi) \in [0,1] \times [0,1]$,

$$\begin{aligned} M(x) \geq{}& \; Z_1(x)^T U_{11} Z_1(x), \\ K_1(x,\xi) ={}& Z_1(x)^T U_{12} Z_2(x,\xi) + Z_2(\xi,x) U_{31} Z_1(\xi) + \int_0^\xi Z_2(\eta,x)^T U_{33} Z_2(\eta,\xi) d\eta \end{aligned}$$

$$+\int_{\xi}^{x} Z_2(\eta,x)^T U_{32} Z_2(\eta,\xi)d\eta + \int_{x}^{1} Z_2(\eta,x)^T U_{22} Z_2(\eta,\xi)d\eta,$$

and

$$K_2(x,\xi) = K_1(\xi,x).$$

Then the operator $\mathcal{P}$*, defined by Equation* (5.12) *is self-adjoint and satisfies*

$$\langle \mathcal{P}w, w\rangle \geq \epsilon \|w\|^2, \text{ for all } w \in L_2(0,1).$$

For completeness, we have provided the proof in Appendix C. A similar proof can be found in [80].

For convenience, we define the set of multipliers and kernels which satisfy Theorem 5.5.

$$\Xi_{\{d_1,d_2,\epsilon\}} = \{M, K_1, K_2 \,:\, M, K_1, K_2 \text{ satisfy the conditions of Theorem 5.5 for } d_1, d_2, \epsilon.\}$$

Note that in Theorem 5.5 we have established only the lower bound for the positive operators. However, we would also require positive operators with known upper bounds. For this purpose, we present the following corollary.

Corollary 5.6. *Given* $d_1, d_2 \in \mathbb{N}$ *and* $\epsilon_1, \epsilon_2 \in \mathbb{R}$ *such that* $0 < \epsilon_1 < \epsilon_2$*, let* $Z_1(x) = Z_{d_1}(x)$ *and* $Z_2(x,\xi) = Z_{d_2}(x,\xi)$ *as defined in Section 2.2. Suppose there exists a matrix* U *such that*

$$U = \begin{bmatrix} U_{11} - \epsilon_1 I_0 & U_{12} & U_{13} \\ \star & U_{22} & U_{23} \\ \star & \star & U_{33} \end{bmatrix} \geq 0,$$

where I_0 is a matrix of zeros of appropriate dimensions except at the 1*-by-*1 *element which has a value of* 1*, and U_{ij} are a partition of U. Additionally,*

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ \star & U_{22} & U_{23} \\ \star & \star & U_{33} \end{bmatrix} \leq \frac{\epsilon_2}{\theta_1 + \theta_2} I,$$

where

$$\theta_1 = \sup_{x \in [0,1]} Z_1(x)^T Z_1(x),$$
$$\theta_2 = \sup_{(x,\xi)\in[0,1]\times[0,1]} \left| \int_0^\xi Z_2(\eta,x)^T Z_2(\eta,\xi) d\eta + \int_x^1 Z_2(\eta,x)^T Z_2(\eta,\xi) d\eta \right|.$$

Let M, K_1 and K_2 be polynomials such that, for $(x,\xi) \in [0,1] \times [0,1]$,

$$\begin{aligned} M(x) =& Z_1(x)^T U_{11} Z_1(x), \\ K_1(x,\xi) =& Z_1(x)^T U_{12} Z_2(x,\xi) + Z_2(\xi,x) U_{31} Z_1(\xi) + \int_0^\xi Z_2(\eta,x)^T U_{33} Z_2(\eta,\xi) d\eta \\ &+ \int_\xi^x Z_2(\eta,x)^T U_{32} Z_2(\eta,\xi) d\eta + \int_x^1 Z_2(\eta,x)^T U_{22} Z_2(\eta,\xi) d\eta, \\ K_2(x,\xi) =& K_1(\xi,x). \end{aligned}$$

Then the operator $\mathcal{P}$, defined by Equation (5.12) *is self-adjoint and satisfies*

$$\epsilon_1 \|y\|^2 \leq \langle \mathcal{P}y, y \rangle \leq \epsilon_2 \|y\|^2, \text{ for all } y \in L_2(0,1).$$

Proof. By substituting ϵ_1 in place of ϵ of Theorem 5.5, it is readily proven that

$$\epsilon_1 \|y\|^2 \leq \langle \mathcal{P}y, y \rangle, \text{ for all } y \in L_2(0,1). \tag{5.13}$$

From the corollary statement,

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ \star & U_{22} & U_{23} \\ \star & \star & U_{33} \end{bmatrix} \leq \frac{\epsilon_2}{\theta_1+\theta_2} I.$$

Thus,

$$\begin{bmatrix} \frac{\epsilon_2}{\theta_1+\theta_2} I - U_{11} & -U_{12} & -U_{13} \\ \star & \frac{\epsilon_2}{\theta_1+\theta_2} I - U_{22} & -U_{23} \\ \star & \star & \frac{\epsilon_2}{\theta_1+\theta_2} I - U_{33} \end{bmatrix} \geq 0,$$

for identity matrices of appropriate dimensions. Thus, using the definitions of M, K_1 and K_2 and the analysis presented in Theorem 5.5, it can be shown that for any $y \in L_2(0,1)$,

$$\int_0^1 y(x) \Bigg(\left[\hat{M}(x) - M(x)\right] + \int_0^x \left[\hat{K}_1(x,\xi) - K_1(x,\xi)\right] y(\xi) d\xi \\ + \int_x^1 \left[\hat{K}_2(x,\xi) - K_2(x,\xi)\right] y(\xi) d\xi \Bigg) dx \geq 0,$$

where

$$\begin{aligned} \hat{M}(x) =& \frac{\epsilon_2}{\theta_1+\theta_2} Z(x)^T Z(x), \\ \hat{K}_1(x,\xi) =& \frac{\epsilon_2}{\theta_1+\theta_2} \left(\int_0^\xi Z_2(\eta,x)^T Z_2(\eta,\xi) d\eta + \int_x^1 Z_2(\eta,x)^T Z_2(\eta,\xi) d\eta \right), \\ \hat{K}_2(x,\xi) =& \frac{\epsilon_2}{\theta_1+\theta_2} \left(\int_0^x Z_2(\eta,x)^T Z_2(\eta,\xi) d\eta + \int_\xi^1 Z_2(\eta,x)^T Z_2(\eta,\xi) d\eta \right). \end{aligned}$$

Thus,

$$\begin{aligned} &\int_0^1 y(x) \left(M(x) y(x) + \int_0^x K_1(x,\xi) y(\xi) d\xi + \int_x^1 K_2(x,\xi) y(\xi) d\xi \right) dx \\ &\quad \leq y(x) \left(\hat{M}(x) y(x) + \int_0^x \hat{K}_1(x,\xi) y(\xi) d\xi + \int_x^1 \hat{K}_2(x,\xi) y(\xi) d\xi \right) dx. \end{aligned}$$

Therefore,

$$\begin{aligned}\langle y, \mathcal{P}y\rangle \leq &\int_0^1 \hat{M}(x)y(x)^2dx + \int_0^1\int_0^x y(x)\hat{K}_1(x,\xi)y(\xi)d\xi dx \\ &+ \int_0^1\int_0^x y(x)\hat{K}_2(x,\xi)y(\xi)d\xi dx \\ \leq &\int_0^1 \hat{M}(x)y(x)^2dx + \int_0^1\int_0^x |y(x)||\hat{K}_1(x,\xi)y(\xi)|d\xi dx \\ &+ \int_0^1\int_0^x |y(x)||\hat{K}_2(x,\xi)||y(\xi)|d\xi dx.\end{aligned}$$

Since $\hat{K}_1(x,\xi) = \hat{K}_2(\xi, x)$, $\hat{K}_1$ and $\hat{K}_2$ have the same supremum over $(x,\xi) \in [0,1] \times [0,1]$. Thus, using the previous equation, we obtain

$$\begin{aligned}\langle y, \mathcal{P}y\rangle \leq &\int_0^1 \hat{M}(x)y(x)^2dx \quad + \int_0^1\int_0^x |y(x)||\hat{K}_1(x,\xi)y(\xi)|d\xi dx \\ &+ \int_0^1\int_0^x |y(x)||\hat{K}_2(x,\xi)||y(\xi)|d\xi dx \\ \leq &\sup_{x\in[0,1]} \hat{M}(x)\int_0^1 y(x)^2dx + \sup_{(x,\xi)\in[0,1]\times[0,1]} |\hat{K}_1(x,\xi)|\int_0^1 |y(x)|dx\int_0^1 |y(\xi)|d\xi.\end{aligned}$$

Using the definitions of θ_1 and θ_2, we obtain

$$\langle y, \mathcal{P}y\rangle \leq \frac{\epsilon_2\theta_1}{\theta_1+\theta_2}\int_0^1 y(x)^2dx + \frac{\epsilon_2\theta_2}{\theta_1+\theta_2}\int_0^1 |y(x)|dx\int_0^1 |y(\xi)|d\xi.$$

Using Proposition B.8 in [81], we obtain

$$\begin{aligned}\langle y, \mathcal{P}y\rangle \leq &\frac{\epsilon_2\theta_1}{\theta_1+\theta_2}\int_0^1 y(x)^2dx + \frac{\epsilon_2\theta_2}{\theta_1+\theta_2}\int_0^1 |y(x)|dx\int_0^1 |y(\xi)|d\xi \\ \leq &\frac{\epsilon_2\theta_1}{\theta_1+\theta_2}\int_0^1 y(x)^2dx + \frac{\epsilon_2\theta_2}{\theta_1+\theta_2}\int_0^1 y(x)^2dx \\ = &\epsilon_2\|y\|^2.\end{aligned}$$

Thus, using Equation (5.13), we conclude that

$$\epsilon_1\|y\|^2 \leq \langle \mathcal{P}y, y\rangle \leq \epsilon_2\|y\|^2, \text{ for all } y \in L_2(0,1).$$

$\square$

For convenience, we define the set of multipliers and kernels which satisfy Corollary 5.6.

$$\begin{aligned}\Omega_{\{d_1,d_2,\epsilon_1,\epsilon_2\}} = \{M, K_1, K_2 \,:\, & M, K_1, K_2 \text{ satisfy the conditions of} \\ & \text{Corollary 5.6 for } d_1, d_2, \epsilon_1, \epsilon_2.\}\end{aligned}$$

5.3 Exponential Stability Analysis

In this section we consider the exponential stability of the system governed by Equations (5.1)-(5.2). The main result depends primarily on the following upper bound - the proof of which can be found in Lemma B.3 in Appendix B.

$$\begin{aligned}\langle \mathcal{A}w, \mathcal{P}w\rangle + \langle w, \mathcal{P}\mathcal{A}w\rangle \leq & \langle w, \mathcal{Q}w\rangle + w_x(1)\int_0^1 Q_3(x)w(x)dx + w_x(0)\int_0^1 Q_4(x)w(x)dx \\ & + w(1)\left(Q_5 w(1) + Q_6 w_x(1) + \int_0^1 Q_7(x)w(x)dx\right) \\ & + w(0)\left(Q_8 w(0) + Q_9 w_x(0) + \int_0^1 Q_{10}(x)w(x)dx\right),\end{aligned}$$

for any $w \in \mathcal{D}_0$, where we define the operator $\mathcal{Q}$ as

$$\left(\mathcal{Q}y\right)(x) = Q_0(x)y(x) + \int_0^x Q_1(x,\xi)y(\xi)d\xi + \int_x^1 Q_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1), \tag{5.14}$$

where

$$\{Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}\} = \mathcal{M}(M, K_1, K_2)$$

and the linear operator $\mathcal{M}$ is defined as follows.

Definition 5.7. *We say*

$$\{Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}\} = \mathcal{M}(M, K_1, K_2)$$

if the following hold

$$Q_0(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(a(x)M(x)) - b(x)M(x)\right) + 2M(x)c(x) - \frac{\alpha\epsilon\pi^2}{2}$$

$$
\begin{aligned}
&+ 2\left[\frac{\partial}{\partial x}\left[a(x)\left(K_1(x,\xi)-K_2(x,\xi)\right)\right]\right]_{\xi=x},\\
Q_1(x,\xi) =&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[a(x)K_1(x,\xi)\right]-b(x)K_1(x,\xi)\right)+c(x)K_1(x,\xi)\\
&+\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left[a(\xi)K_1(x,\xi)\right]-b(\xi)K_1(x,\xi)\right)+c(\xi)K_1(x,\xi),\\
Q_2(x,\xi) =&Q_1(\xi,x),\\
Q_3(x) =&2n_5a(1)K_1(1,x),\\
Q_4(x) =&-2n_2a(0)K_2(0,x),\\
Q_5 =&2n_6n_4a(1)M(1)-n_6^2\left[a_x(1)M(1)+a(1)M_x(1)-b(1)M(1)\right],\\
Q_6 =&2n_6n_5a(1)M(1),\\
Q_7(x) =&K_1(1,x)\left[2n_4a(1)+2n_6b(1)\right]-2n_6\left[a_x(1)K_1(1,x)+a(1)K_{1,x}(1,x)\right],\\
Q_8 =&-2n_3n_1a(0)M(0)+n_3^2\left[a_x(0)M(0)+a(0)M_x(0)-b(0)M(0)+\frac{\alpha\epsilon\pi^2}{2}\right],\\
Q_9 =&-2n_3n_2a(0)M(0),\\
Q_{10}(x) =&-K_2(0,x)\left[2n_1a(0)+2n_3b(0)\right]+2n_3\left[a_x(0)K_2(0,x)+a(0)K_{2,x}(0,x)\right],
\end{aligned}
$$

where $K_{1,x}(1,x)=\left[K_{1,x}(x,\xi)|_{x=1}\right]_{\xi=x}$, $K_{2,x}(0,x)=\left[K_{2,x}(x,\xi)|_{x=0}\right]_{\xi=x}$ *and* $\epsilon>0$ *and* n_i, $i\in\{1,\cdots,6\}$, *are scalars.*

We now present the theorem for exponential stability analysis.

Theorem 5.8. *Suppose that there exist scalars* $\epsilon,\delta>0$ *and* $\{M,K_1,K_2\}\in\Xi_{d_1,d_2,\epsilon}$ *such that*

$$
\begin{aligned}
&\{-Q_0-2\delta M,-Q_1-2\delta K_1,-Q_2-2\delta K_2\}\in\Xi_{d_1,d_2,0},\\
&Q_3=Q_4=Q_6=Q_7=Q_9=Q_{10}=0,\\
&Q_5\le 0,\quad Q_8\le 0,\ \textit{for all } n_j, j\in\{1,\cdots,6\},
\end{aligned}
$$

where n_j *are given by Definition 5.1 and*

$$\{Q_0,Q_1,Q_2,Q_3,Q_4,Q_5,Q_6,Q_7,Q_8,Q_9,Q_{10}\}=\mathcal{M}(M,K_1,K_2).$$

Then, for any initial condition $w_0 \in \mathcal{D}_0$*, there exists a scalar* $M \geq 0$ *such that the classical solution* $w(x,t)$ *of Equations* (5.1)-(5.2) *satisfies*

$$\|w(\cdot,t)\| \leq e^{-\delta t} M, \quad t > 0.$$

For $w_0 \in L_2(0,1)$*, the same result holds for the weak solution.*

Proof. Consider the following Lyapunov function $V(w(\cdot,t)) = \langle w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle$, where

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1).$$

Taking the derivative along trajectories of the system, we have

$$\begin{aligned}\frac{d}{dt}V(w(\cdot,t)) &= \langle w_t(\cdot,t), (\mathcal{P}w(\cdot,t))\rangle + \langle w(\cdot,t), (\mathcal{P}w_t(\cdot,t))\rangle \\ &= \langle \mathcal{A}w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle + \langle w(\cdot,t), \mathcal{P}\mathcal{A}w(\cdot,t)\rangle .\end{aligned}$$

Since the initial condition $w_0 \in \mathcal{D}_0$, from Lemma 5.4, the classical solution $w(\cdot,t) \in \mathcal{D}_0$ exists for all $t \geq 0$. For $\mathcal{P}$ as defined in (5.12) and $\mathcal{M}$ as defined in Definition 5.7, it is shown in Appendix B that if

$$\{Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}\} = \mathcal{M}(M, K_1, K_2),$$

then

$$\begin{aligned}\frac{d}{dt}V(w(\cdot,t)) =& \langle \mathcal{A}w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle + \langle w(\cdot,t), \mathcal{P}\mathcal{A}w(\cdot,t)\rangle \\ \leq& \langle w(\cdot,t), \mathcal{Q}w(\cdot,t)\rangle \\ &+ w_x(1,t)\int_0^1 Q_3(x)w(x,t)dx + w_x(0,t)\int_0^1 Q_4(x)w(x,t)dx \\ &+ w(1,t)\left(Q_5 w(1,t) + Q_6 w_x(1,t) + \int_0^1 Q_7(x)w(x,t)dx\right) \\ &+ w(0,t)\left(Q_8 w(0,t) + Q_9 w_x(0,t) + \int_0^1 Q_{10}(x)w(x,t)dx\right),\end{aligned}$$

where the operator $\mathcal{Q}$ is defined in Equation (5.14). Now, since by assumption $Q_3 = Q_4 = Q_6 = Q_7 = Q_9 = Q_{10} = 0$, $Q_5 \leq 0$ and $Q_8 \leq 0$, we have

$$\begin{aligned}\frac{d}{dt}V(w(\cdot,t)) &\leq \langle w(\cdot,t), \mathcal{Q}w(\cdot,t)\rangle \\ &= \int_0^1 w(x,t)\Big(Q_0(x)w(x,t) + \int_0^x Q_1(x,\xi)w(\xi,t)d\xi \\ &\qquad + \int_x^1 Q_2(x,\xi)w(\xi,t)d\xi\Big)dx \\ &= \langle w(\cdot,t), \mathcal{Q}w(\cdot,t)\rangle . \end{aligned} \tag{5.15}$$

Since

$$\{-Q_0 - 2\delta M, -Q_1 - 2\delta K_1, -Q_2 - 2\delta K_2\} \in \Xi_{d_1,d_2,0},$$

we have that

$$\begin{aligned}&\int_0^1 Q_0(x)w(x,t)^2dx + \int_0^1 w(x,t)\left(\int_0^x Q_1(x,\xi)w(\xi,t)d\xi + \int_0^x Q_2(x,\xi)w(\xi,t)d\xi\right)dx \\ &\leq -2\delta\int_0^1 M(x)w(x,t)^2dx \\ &\quad - 2\delta\int_0^1 w(x,t)\left(\int_0^x K_1(x,\xi)w(\xi,t)d\xi + \int_0^x K_2(x,\xi)w(\xi,t)d\xi\right)dx.\end{aligned}$$

Using the definitions of operators $\mathcal{P}$ and $\mathcal{Q}$

$$\langle w(\cdot,t), \mathcal{Q}w(\cdot,t)\rangle \leq -2\delta\,\langle w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle .$$

Substituting into Equation (5.15) produces

$$\frac{d}{dt}V(w(\cdot,t)) \leq \langle w(\cdot,t), \mathcal{Q}w(\cdot,t)\rangle \leq -2\delta\,\langle w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle .$$

Hence we conclude that

$$\frac{d}{dt}V(w(\cdot,t)) \leq -2\delta V(w(\cdot,t)), \quad t > 0.$$

Integrating in time yields $\langle w(\cdot,t), (\mathcal{P}w)(\cdot,t)\rangle \leq e^{-2\delta t}\langle w_0, \mathcal{P}w_0\rangle$ and since, $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$, we have

$$\epsilon\|w(\cdot,t)\|^2 \leq \langle w(\cdot,t), (\mathcal{P}w)(\cdot,t)\rangle \leq e^{-2\delta t}\langle w_0, \mathcal{P}w_0\rangle, \quad t > 0$$

which implies

$$\|w(\cdot,t)\| \le e^{-\delta t}\sqrt{\frac{\langle w_0, \mathcal{P} w_0\rangle}{\epsilon}}, \quad t>0.$$

Setting

$$M = \sqrt{\frac{\langle w_0, \mathcal{P} w_0\rangle}{\epsilon}}$$

completes the proof. □

5.3.1 Numerical Results. To illustrate the accuracy of the the stability test, we perform the following numerical experiments. We consider the following two parabolic PDEs:

$$\begin{aligned} w_t(x,t) =& w_{xx}(x,t) + \lambda w(x,t), \text{ and} && (5.16)\\ w_t(x,t) =& \left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t) \\ &+ \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(x,t), && (5.17)\end{aligned}$$

where λ is a scalar which may be chosen freely. We consider the following boundary conditions for these two equations:

$$\text{Dirichlet:} = w(0) = 0, \quad w(1) = 0, \tag{5.18}$$

$$\text{Neumann:} = w_x(0) = 0, \quad w_x(1) = 0, \tag{5.19}$$

$$\text{Mixed:} = w(0) = 0, \quad w_x(1) = 0, \tag{5.20}$$

$$\text{Robin:} = w(0) = 0, \quad w(1) + w_x(1) = 0. \tag{5.21}$$

Table 5.1 illustrates the maximum λ for which we can construct a Lyapunov function for Equation (5.16) using the analysis presented in Theorem 5.8 as a function of the degree of polynomial representation $d_1 = d_2 = d$ with $\epsilon = \delta = 0.001$.

Table 5.1. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which a Lyapunov function proving the exponential stability of $w_t = w_{xx} + \lambda w$ can be constructed using Theorem 5.8

Boundary Conditions	$d = 3$	4	5	6	7
Dirichlet					
$w(0) = 0, w(1) = 0$	$\lambda = 0.78$	3.67	6.14	9.63	9.83
Neumann					
$w_x(0) = 0, w_x(1) = 0$	-0.0061	-0.002	-0.002	-0.002	-0.002
Mixed					
$w(0) = 0, w_x(1) = 0$	0.7263	2.4353	2.4567	2.4597	2.4597
Robin					
$w(0) = 0, w(1) + w_x(1) = 0$	0.7843	3.9124	4.095	4.10	4.10

Table 5.2 presents a comparison of the maximum λ as calculated by Theorem 5.8 and the maximum λ calculated using Sturm-Liouville theory presented in Table E.1 in Appendix E.

Table 5.2. Comparison of maximum λ for which a Lyapunov function proving the exponential stability of $w_t = w_{xx} + \lambda w$ can be constructed using Theorem 5.8 and maximum λ predicted by Sturm-Liouville theory for stability.

Boundary Conditions	Maximum λ using Theorem 5.8	Maximum λ using Sturm Liouville theory
Dirichlet		
$w(0) = 0, w(1) = 0$	9.83	$\pi^2 \approx 9.86$
Neumann		
$w_x(0) = 0, w_x(1) = 0$	-0.002	0
Mixed		
$w(0) = 0, w_x(1) = 0$	2.4597	$\pi^2/4 \approx 2.47$
Robin		
$w(0) = 0, w(1) + w_x(1) = 0$	4.10	4.12

Table 5.3 illustrates the maximum λ for which we can construct a Lyapunov function for Equation (5.17) using the analysis presented in Theorem 5.8 as a function of $d_1 = d_2 = d$ with the previously chosen parameters of $\epsilon = \delta = 0.001$.

Table 5.3. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which a Lyapunov function proving the exponential stability of Equation (5.17) can be constructed using Theorem 5.8.

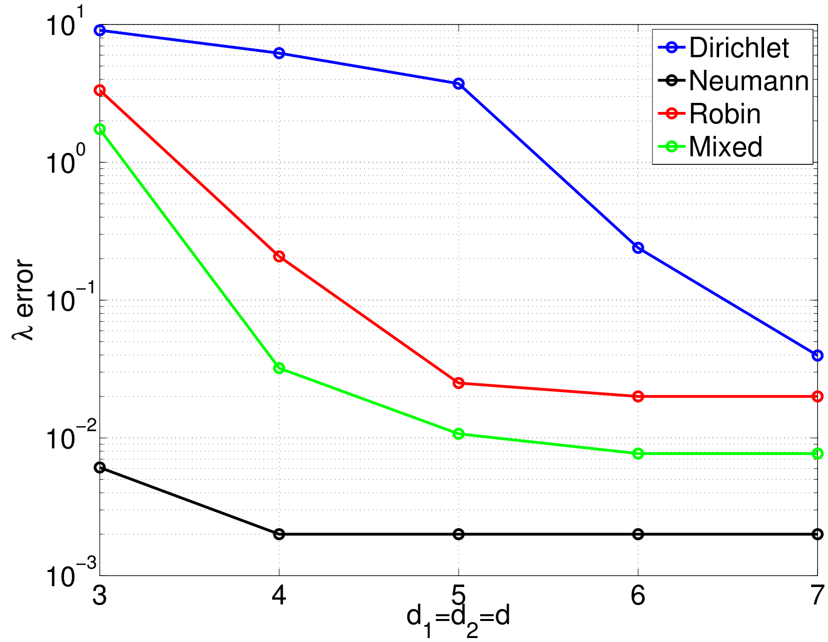
Boundary Conditions	$d = 3$	4	5	6	7
Dirichlet					
$w(0) = 0, w(1) = 0$	$\lambda = 99.9$	15.615	18.837	18.853	18.87
Neumann					
$w_x(0) = 0, w_x(1) = 0$	-99.9	-0.2625	-0.2625	-0.2625	-0.2625
Mixed					
$w(0) = 0, w_x(1) = 0$	4.37	4.61	4.61	4.62	4.62
Robin					
$w(0) = 0, w(1) + w_x(1) = 0$	7.89	7.89	7.89	7.89	7.91

Table 5.4 presents a comparison of the maximum λ as calculated by Theorem 5.8 and the maximum λ calculated using finite-difference approach presented in Table E.2 in Appendix E.

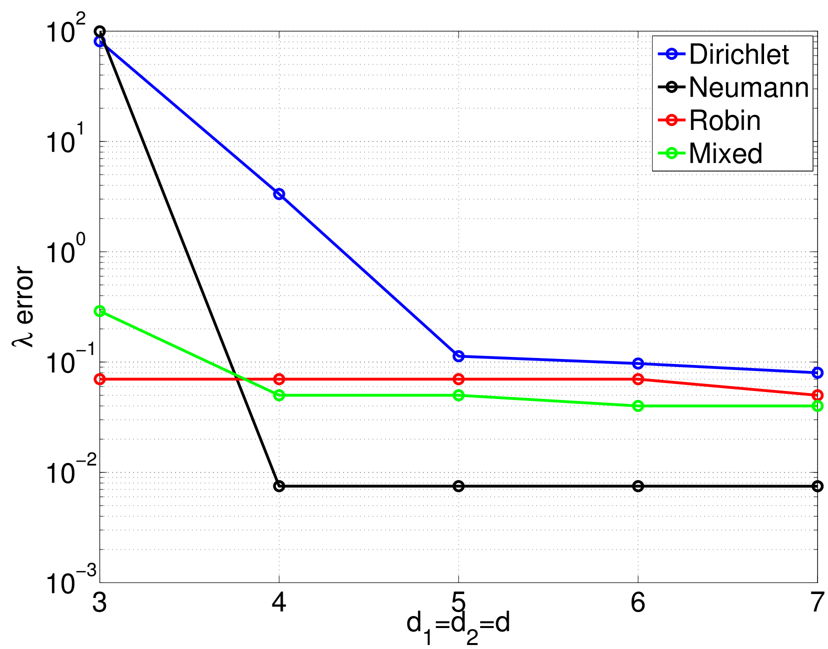
Table 5.4. Comparison of maximum λ for which a Lyapunov function proving the exponential stability of Equation (5.17) can be constructed using Theorem 5.8 and maximum λ predicted by finite-difference approach.

Boundary Conditions	Maximum λ using Theorem 5.8	Maximum λ using finite-differences
Dirichlet		
$w(0) = 0, w(1) = 0$	18.87	18.95
Neumann		
$w_x(0) = 0, w_x(1) = 0$	-0.2625	-0.255
Mixed		
$w(0) = 0, w_x(1) = 0$	4.62	4.66
Robin		
$w(0) = 0, w(1) + w_x(1) = 0$	7.91	7.96

To illustrate the accuracy of the proposed stability analysis methodology, we plot the error between the calculated maximum stable λ using Theorem 5.8 versus the calculated/estimated maximum stable λ for Equations (5.16) and (5.17). Figures 5.1(a)-5.1(b) provide these results. As is evident, for degree $d_1 = d_2 = 7$ the difference between the calculated and predicted maximum λ is less that 0.1. Thus, we conclude that the provided methodology is quite accurate in analyzing the stability of the parabolic PDEs considered.



(a) Equation (5.16).



(b) Equation (5.17).

Figure 5.1. Error between calculated max. λ using Theorem 5.8 and calculated/estimated max. λ using Sturm-Liouville/finite-differences.

CHAPTER 6

STATE FEEDBACK BASED BOUNDARY CONTROL OF PARABOLIC PDES

In this chapter we consider controller synthesis for parabolic PDEs. Similar to Chapter 5, we accomplish this task by constructing Lyapunov functions parametrized by sum-of-squares polynomials. In addition, the controllers are parametrized by polynomials.

We consider Equations (5.1)- (5.2), given in Chapter 5, with inhomogeneous boundary conditions given by

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t), \quad x \in [0,1], \quad t \geq 0, \tag{6.1}$$

with boundary conditions of the form

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{and} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t). \tag{6.2}$$

Here, the real valued function $u(t) \in \mathbb{R}$ is called the *control input*. In addition, recall the properties of the system, namely, the functions a, b and c are polynomial functions in x. Moreover, the function a satisfies

$$a(x) \geq \alpha > 0, \quad \text{for} \quad x \in [0,1]. \tag{6.3}$$

The scalars $\nu_i, \rho_j \in \mathbb{R}$, $i,j \in \{1,2\}$ satisfy

$$|\nu_1| + |\nu_2| > 0 \quad \text{and} \quad |\rho_1| + |\rho_2| > 0. \tag{6.4}$$

We wish to design a controller $\mathcal{F} : H^2(0,1) \to \mathbb{R}$ such that if

$$u(t) = \mathcal{F}w(\cdot,t), \tag{6.5}$$

then the system given by Equations (6.1)-(6.2) is stable. We also assume that access to the complete state is available for the design of controllers. Such type of controllers are called *full state feedback based controllers*.

For PDEs in the form of Equations (6.1)-(6.2), we define the following first order form

$$\dot{\mathbf{w}}(t) = \mathcal{A}\mathbf{w}(t), \quad \mathbf{w} \in \mathcal{D} \tag{6.6}$$

where the operator $\mathcal{A} : H^2(0,1) \to L_2(0,1)$ is defined in Equation (5.6) as

$$\left(\mathcal{A}y\right)(x) = a(x)y_{xx}(x) + b(x)y_x(x) + c(x)y(x), \tag{6.7}$$

and

$$\mathcal{D} = \{y \in H^2(0,1) : \quad \nu_1 y(0) + \nu_2 y_x(0) = 0 \text{ and } \rho_1 y(1) + \rho_2 y_x(1) = \mathcal{F}y\}. \tag{6.8}$$

If the operator $\mathcal{F}$ is of the form $\mathcal{F}y = R_1 y(1) + R_2 y_x(1)$, $y \in H^2(0,1)$, then, using the analysis presented in Section 5.1 the uniqueness and existence of classical (weak) solutions of Equation (6.6), and hence Equations (6.1)-(6.2), can be established. However, for a more general form of operator $\mathcal{F}$ which we consider, it is considerably more difficult to establish the uniqueness and existence of solutions. Thus, we make the following assumption:

Assumption 6.1. *For any operator $\mathcal{F} : H^2(0,1) \to \mathbb{R}$ and initial condition $w_0 \in \mathcal{D}$, there exists a classical solution to Equations* (6.1)-(6.2) *with $u(t)$ given by Equation* (6.5). *Similarly, for any initial condition $w_0 \in L_2(0,1)$, there exists a weak solution to Equations* (6.1)-(6.2).

For later use, we present the following definition.

Definition 6.2. *Given scalars ν_1, ν_2, ρ_1 and ρ_2, we define*

$$\{m_1, m_2, m_3\} = \begin{cases} \{-\frac{\nu_1}{\nu_2}, 0, 1\} & \text{if } \nu_1, \nu_2 \neq 0 \\ \{0, 1, 0\} & \text{if } \nu_1 \neq 0, \nu_2 = 0 \\ \{0, 0, 1\} & \text{if } \nu_1 = 0, \nu_2 \neq 0. \end{cases}$$

With this definition, the boundary conditions given in Equation (6.2) *can be represented as*

$$w_x(0,t) = m_1 w(0,t) + m_2 w_x(0,t), \quad w(0) = m_3 w(0,t).$$

6.1 Exponentially Stabilizing Boundary Control

In this section we consider the synthesis of controller $\mathcal{F}$ such that if the control input

$$u(t) = \mathcal{F} w(\cdot,t),$$

then, the system governed by Equations (6.1)-(6.2) is exponentially stable. The main result depends primarily on the following upper bound - the proof of which can be found in Lemma B.7 in Appendix B.

$$\begin{aligned}
&\langle \mathcal{A}\mathcal{P}z(\cdot,t), z(\cdot,t)\rangle + \langle z(\cdot,t), \mathcal{P}\mathcal{A}z(\cdot,t)\rangle \\
&\leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle \\
&\quad + z(0,t)\left(T_3 z(0,t) + \int_0^1 T_4(x) z(x,t)dx\right) + z_x(0,t)\int_0^1 T_5(x) z(x,t)dx \\
&\quad + \int_0^1 \frac{1}{M(0)} T_6(x) z(x,t)dx \left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2\right) z(0,t) \\
&\quad + \int_0^1 \frac{1}{M(0)} T_6(x) z(x,t)dx \int_0^1 \alpha\epsilon\pi^2 z(x,t)dx \\
&\quad + z(1,t)\left(T_7 z(1,t) + T_8 z_x(1,t)\right),
\end{aligned}$$

where $z(\cdot,t) = \mathcal{P}^{-1}w(\cdot,t)$, w being a solution of Equations (6.1)-(6.2),

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1),$$

and we define the operator $\mathcal{T}$ as

$$(\mathcal{T}y)(x) = T_0(x)y(x) + \int_0^x T_1(x,\xi)y(\xi)d\xi + \int_x^1 T_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1), \quad (6.9)$$

where

$$\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \mathcal{N}(M, K_1, K_2)$$

and the linear operator $\mathcal{N}$ is defined as follows.

Definition 6.3. *We say*

$$\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \mathcal{N}(M, K_1, K_2)$$

if the following hold

$$\begin{aligned}
T_0(x) =& a_{xx}(x)M(x) + a(x)M_{xx}(x) - b_x(x)M(x) + b(x)M_x(x) + 2c(x)M(x) \\
&+ 2a(x)\left[K_{1,x}(x,x) - K_{2,x}(x,x)\right] - \frac{\pi^2\alpha\epsilon}{2}, \\
T_1(x,\xi) =& \left[a(x)K_{1,xx}(x,\xi) + a(\xi)K_{1,\xi\xi}(x,\xi)\right] + \left[b(x)K_{1,x}(x,\xi) + b(\xi)K_{1,\xi}(x,\xi)\right] \\
&+ \left[c(x)K_1(x,\xi) + c(\xi)K_1(x,\xi)\right], \\
T_2(x,\xi) =& \left[a(x)K_{2,xx}(x,\xi) + a(\xi)K_{2,\xi\xi}(x,\xi)\right] + \left[b(x)K_{2,x}(x,\xi) + b(\xi)K_{2,\xi}(x,\xi)\right] \\
&+ \left[c(x)K_2(x,\xi) + c(\xi)K_2(x,\xi)\right], \\
T_3 =& - m_3\left(a(0)M_x(0) - \frac{1}{2}\alpha\epsilon\pi^2\right) + m_3\left(a_x(0) - b(0)\right)M(0) \\
&- 2a(0)\left(m_1 M(0) + (m_2 - 1)M_x(0)\right), \\
T_4 =& (m_3 - 1)(a_x(0) - b(0))K_2(0,x) \\
&- 2a(0)\left[(m_2 - 1)K_{2,x}(0,x) + m_1 K_2(0,x)\right], \\
T_5(x) =& - 2m_2(m_3 - 1)a(0)K_2(0,x), \\
T_6(x) =& 2(m_3 - 1)K_2(0,x), \\
T_7 =& - a_x(1)M(1) + a(1)M_x(1) + b(1)M(1), \\
T_8 =& 2a(1)M(1),
\end{aligned}$$

where $K_{1,x}(1,x) = \left[K_{1,x}(x,\xi)|_{x=1}\right]_{\xi=x}$, $K_{2,x}(0,x) = \left[K_{2,x}(x,\xi)|_{x=0}\right]_{\xi=x}$ *and* $\epsilon > 0$ *and* m_i, $i \in \{1, \cdots, 3\}$, *are scalars.*

We present the following theorem.

Theorem 6.4. *Suppose that there exist scalars* $\epsilon, \delta > 0$ *and* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ *such that*

$$\{-T_0 - 2\delta M, -T_1 - 2\delta K_1, -T_2 - 2\delta K_2\} \in \Xi_{d_1,d_2,0},$$
$$T_3 \leq 0, \quad T_4(x) = T_5(x) = T_6(x) = 0,$$

for all m_j*,* $j \in \{1, \cdots, 3\}$ *where* m_j *are given by Definition 6.2 and*

$$\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \mathcal{N}(M, K_1, K_2).$$

Define the operator $\mathcal{F} := \mathcal{Z}\mathcal{P}^{-1}$ *where, for any* $y \in H^2(0,1)$*,*

$$\mathcal{Z}y = \begin{cases} Z_1 y(1) + \int_0^1 Z_2(x) y(x) dx & \rho_1 = 0, \rho_2 \neq 0 \\ Z_3 y_x(1) + \int_0^1 Z_4(x) y(x) dx & \rho_1 \neq 0, \rho_2 = 0 \\ Z_5 y(1) + \int_0^1 Z_6(x) y(x) dx & \rho_1 \neq 0, \rho_2 \neq 0 \end{cases} \cdot$$

Here, Z_1*,* Z_3 *and* Z_5 *are any scalars that satisfy*

$$Z_1 < 0 \quad \text{and} \quad Z_1 < -\frac{\rho_2}{2a(1)}\left(T_7 - 2a(1)M_x(1)\right),$$
$$Z_3 < 0 \quad \text{and} \quad \frac{1}{Z_3} < -\frac{1}{\rho_1 M(1)}\frac{T_7}{T_8},$$
$$Z_5 < 0 \quad \text{and} \quad Z_5 < -\frac{\rho_2}{2a(1)}\left(T_7 - \frac{\rho_1}{\rho_2}T_8 - 2a(1)M_x(1)\right),$$

and polynomials $Z_2(x)$*,* $Z_4(x)$ *and* $Z_6(x)$ *are defined as*

$$Z_2(x) = \rho_2 K_{1,x}(1,x),$$
$$Z_4(x) = \rho_1 K_1(1,x),$$
$$Z_6(x) = \rho_2\left(\frac{\rho_1}{\rho_2}K_1(1,x) + K_{1,x}(1,x)\right).$$

Additionally,

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1).$$

Then for any solution w of (6.1) - (6.2) *with* $u(t) = \mathcal{F}w(\cdot, t)$ *and initial condition* $w_0 \in \mathcal{D}$ *there exists a scalar* $M \geq 0$ *such that*

$$\|w(\cdot, t)\| \leq e^{-\delta t} M, \quad t > 0.$$

Proof. Consider the following Lyapunov function $V(w(\cdot, t)) = \langle w(\cdot, t), \mathcal{P}^{-1} w(\cdot, t)\rangle$. Note that this Lyapunov functional is well-defined because from Assumption 6.1, the solution (unique or weak) exists. Moreover, the bounded linear operator $\mathcal{P}$ is strictly positive. Thus, its inverse $\mathcal{P}^{-1}$ exists and is bounded and linear [35].

Taking the time derivative along trajectories of the system, we have

$$\frac{d}{dt} V(w(\cdot, t)) = \left\langle \mathcal{A}w(t), \mathcal{P}^{-1} w(t)\right\rangle + \left\langle \mathcal{P}^{-1} w(t), \mathcal{A}w(t)\right\rangle,$$

where we have used the fact that $\mathcal{P} = \mathcal{P}^\star$ implies $\mathcal{P}^{-1} = (\mathcal{P}^\star)^{-1}$. Now let $z = \mathcal{P}^{-1} w$. Then

$$\begin{aligned}\frac{d}{dt} V(w(\cdot, t)) &= \left\langle \mathcal{A}\mathcal{P}\mathcal{P}^{-1} w(\cdot, t), \mathcal{P}^{-1} w(\cdot, t)\right\rangle + \left\langle \mathcal{P}^{-1} w(\cdot, t), \mathcal{A}\mathcal{P}\mathcal{P}^{-1} w(\cdot, t)\right\rangle \\ &= \langle \mathcal{A}\mathcal{P} z(\cdot, t), z(\cdot, t)\rangle + \langle z(\cdot, t), \mathcal{A}\mathcal{P} z(\cdot, t)\rangle .\end{aligned}$$

From Lemma B.7,

$$\begin{aligned}&\frac{d}{dt} V(w(\cdot, t)) \\ &= \langle \mathcal{A}\mathcal{P} z(\cdot, t), z(\cdot, t)\rangle + \langle z(\cdot, t), \mathcal{A}\mathcal{P} z(\cdot, t)\rangle \\ &\leq \langle z(\cdot, t), \mathcal{T} z(\cdot, t)\rangle \\ &\quad + z(0, t) \left(T_3 z(0, t) + \int_0^1 T_4(x) z(x, t) dx \right) + z_x(0, t) \int_0^1 T_5(x) z(x, t) dx \\ &\quad + \int_0^1 \frac{1}{M(0)} T_6(x) z(x, t) dx \left(-a(0) M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2 \right) z(0, t) \\ &\quad + \int_0^1 \frac{1}{M(0)} T_6(x) z(x, t) dx \int_0^1 \alpha\epsilon\pi^2 z(x, t) dx \\ &\quad + z(1, t) \left(T_7 z(1, t) + T_8 z_x(1, t) \right),\end{aligned}$$

where the operator $\mathcal{T}$ is defined in Equation (6.9). From the theorem statement we have that $T_4(x) = T_5(x) = T_6(x) = 0$ and $T_3 \leq 0$, thus

$$\begin{aligned}
&\frac{d}{dt}V(w(\cdot,t)) \\
&= \langle \mathcal{AP}z(\cdot,t), z(\cdot,t)\rangle + \langle z(\cdot,t), \mathcal{AP}z(\cdot,t)\rangle \\
&\leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle + z(1,t)\left(T_7 z(1,t) + T_8 z_x(1,t)\right).
\end{aligned} \tag{6.10}$$

From Equation (6.4),

$$|\rho_1| + |\rho_2| > 0.$$

Thus, there are three cases possible,

$$\rho_1 = 0 \text{ and } \rho_2 \neq 0, \quad \rho_1 \neq 0 \text{ and } \rho_2 = 0, \quad \rho_1 \neq 0 \text{ and } \rho_2 \neq 0.$$

For the case when $\rho_1 = 0$ and $\rho_2 \neq 0$,

$$\rho_2 w_x(1,t) = u(t) = \mathcal{F}w(\cdot,t) = \mathcal{FPP}^{-1}w(\cdot,t) = \mathcal{Z}z(\cdot,t),$$

hence

$$w_x(1,t) = \frac{1}{\rho_2}\mathcal{Z}z(\cdot,t).$$

Since, $w = \mathcal{P}z$, we have

$$\begin{aligned}
w_x(1,t) &= \frac{1}{\rho_2}\mathcal{Z}z(\cdot,t) \\
&= M_x(1)z(1,t) + M(1)z_x(1,t) + \int_0^1 K_{1,x}(1,x)z(x,t)dx.
\end{aligned}$$

Hence,

$$M(1)z_x(1,t) = \frac{1}{\rho_2}\mathcal{Z}z(\cdot,t) - M_x(1)z(1,t) - \int_0^1 K_{1,x}(1,x)z(x,t)dx.$$

Multiplying both sides by $2a(1)$,

$$T_8 z_x(1,t) = \frac{2a(1)}{\rho_2}\mathcal{Z}z(\cdot,t) - 2a(1)M_x(1)z(1,t)$$

$$- \int_0^1 2a(1) K_{1,x}(1,x) z(x,t) dx.$$

Substituting in Equation (6.10),

$$\begin{aligned}
&\frac{d}{dt} V(w(\cdot,t)) \\
&= \langle \mathcal{A}\mathcal{P} z(\cdot,t), z(\cdot,t) \rangle + \langle z(\cdot,t), \mathcal{A}\mathcal{P} z(\cdot,t) \rangle \\
&\leq \langle z(\cdot,t), \mathcal{T} z(\cdot,t) \rangle + z(1,t) \frac{2a(1)}{\rho_2} \mathcal{Z} z(\cdot,t) \\
&\quad + z(1,t) \left(\left(T_7 - 2a(1) M_x(1) \right) z(1,t) - \int_0^1 2a(1) K_{1,x}(1,x) z(x,t) dx \right).
\end{aligned}$$

Using the definition of $\mathcal{Z}$ from the theorem statement

$$\begin{aligned}
&\frac{d}{dt} V(w(\cdot,t)) \\
&= \langle \mathcal{A}\mathcal{P} z(\cdot,t), z(\cdot,t) \rangle + \langle z(\cdot,t), \mathcal{A}\mathcal{P} z(\cdot,t) \rangle \\
&\leq \langle z(\cdot,t), \mathcal{T} z(\cdot,t) \rangle + z(1,t)^2 \left(T_7 - 2a(1) M_x(1) + \frac{2a(1)}{\rho_2} Z_1 \right).
\end{aligned}$$

Since Z_1 is any scalar that satisfies

$$Z_1 < 0 \quad \text{and} \quad Z_1 < -\frac{\rho_2}{2a(1)} \left(T_7 - 2a(1) M_x(1) \right),$$

there exists a scalar $\zeta_1 > 0$ such that

$$T_7 - 2a(1) M_x(1) + \frac{2a(1)}{\rho_2} Z_1 = -\zeta_1.$$

Thus, for the case when $\rho_1 = 0$ and $\rho_2 \neq 0$ we get that there exists a scalar $\zeta_1 > 0$ such that

$$\frac{d}{dt} V(w(\cdot,t)) \leq \langle z(\cdot,t), \mathcal{T} z(\cdot,t) \rangle - \zeta_1 z(1,t)^2. \tag{6.11}$$

For the case when $\rho_1 \neq 0$ and $\rho_2 = 0$,

$$\rho_1 w(1,t) = u(t) = \mathcal{F} w(\cdot,t) = \mathcal{F}\mathcal{P}\mathcal{P}^{-1} w(\cdot,t) = \mathcal{Z} z(\cdot,t),$$

hence

$$w(1,t) = \frac{1}{\rho_1}\mathcal{Z}z(\cdot,t).$$

Using the fact that $w = \mathcal{P}z$ we obtain

$$w(1,t) = \frac{1}{\rho_1}\mathcal{Z}z(\cdot,t) = M(1)z(1,t) + \int_0^1 K_1(1,x)z(x,t)dx.$$

Now, by definition,

$$\mathcal{Z}z(\cdot,t) = Z_3 z_x(1,t) + \int_0^1 Z_4(x)z(x,t)dx.$$

Combining the last two statements and using the definition of $Z_4(x)$,

$$z_x(1,t) = \frac{\rho_1}{Z_3}M(1)z(1,t).$$

Note that this is well defined since $Z_3 < 0$. Substituting in Equation (6.10)

$$\begin{aligned}
&\frac{d}{dt}V(w(\cdot,t)) \\
&= \langle \mathcal{A}\mathcal{P}z(\cdot,t), z(\cdot,t)\rangle + \langle z(\cdot,t), \mathcal{A}\mathcal{P}z(\cdot,t)\rangle \\
&\leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle + z(1,t)^2\left(T_7 + \frac{\rho_1}{Z_3}M(1)T_8\right).
\end{aligned}$$

Since, from the theorem statement,

$$Z_3 < 0 \quad \text{and} \frac{1}{Z_3} < -\frac{1}{\rho_1 M(1)}\frac{T_7}{T_8},$$

there exists a scalar $\zeta_2 > 0$ such that

$$T_7 + \frac{\rho_1}{Z_3}M(1)T_8 = -\zeta_2,$$

where we have used the fact that $T_8 = 2a(1)M(1) > 0$. Hence, for the case when $\rho_1 \neq 0$ and $\rho_2 = 0$, there exists a scalar $\zeta_2 > 0$ such that

$$\frac{d}{dt}V(w(\cdot,t)) \leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle - \zeta_2 z(1,t)^2. \tag{6.12}$$

For the case when $\rho_1 \neq 0$ and $\rho_2 \neq 0$,

$$\rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t) = \mathcal{F}w(\cdot,t) = \mathcal{F}\mathcal{P}\mathcal{P}^{-1}w(\cdot,t) = \mathcal{Z}z(\cdot,t),$$

hence using $w = \mathcal{P}z$

$$M(1)z_x(1,t) = \frac{1}{\rho_2}\mathcal{Z}z(\cdot,t) - \frac{\rho_1}{\rho_2}M(1)z(1,t) - M_x(1)z(1,t) - \frac{\rho_1}{\rho_2}\int_0^1 K_1(1,x)z(x,t)dx - \int_0^1 K_{1,x}(1,x)z(x,t)dx.$$

Multiplying both sides by $2a(1)$

$$\begin{aligned} &T_8 z_x(1,t) \\ &= \frac{2a(1)}{\rho_2}\mathcal{Z}z(\cdot,t) - \frac{\rho_1}{\rho_2}T_8 z(1,t) - 2a(1)M_x(1)z(1,t) \\ &\quad - 2a(1)\frac{\rho_1}{\rho_2}\int_0^1 K_1(1,x)z(x,t)dx - 2a(1)\int_0^1 K_{1,x}(1,x)z(x,t)dx. \end{aligned}$$

Substituting in Equation (6.10) we obtain

$$\begin{aligned} &\frac{d}{dt}V(w(\cdot,t)) \\ &= \langle \mathcal{A}\mathcal{P}z(\cdot,t), z(\cdot,t)\rangle + \langle z(\cdot,t), \mathcal{A}\mathcal{P}z(\cdot,t)\rangle \\ &\leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle + z(1,t)\frac{2a(1)}{\rho_2}\mathcal{Z}z(\cdot,t) \\ &\quad + z(1,t)^2\left[T_7 - \frac{\rho_1}{\rho_2}T_8 - 2a(1)M_x(1)\right] \\ &\quad - z(1,t)\int_0^1 2a(1)\left(\frac{\rho_1}{\rho_2}K_1(1,x) + K_{1,x}(1,x)\right)z(x,t)dx. \end{aligned}$$

Using the definition of $\mathcal{Z}$ from the theorem statement for the case when $\rho_1 \neq 0$ and $\rho_2 \neq 0$ we obtain

$$\begin{aligned} &\frac{d}{dt}V(w(\cdot,t)) \\ &= \langle \mathcal{A}\mathcal{P}z(\cdot,t), z(\cdot,t)\rangle + \langle z(\cdot,t), \mathcal{A}\mathcal{P}z(\cdot,t)\rangle \\ &\leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle \\ &\quad + z(1,t)^2\left(T_7 - \frac{\rho_1}{\rho_2}T_8 - 2a(1)M_x(1) + \frac{2a(1)}{\rho_2}Z_5\right). \end{aligned}$$

Since, by definition, Z_5 is any scalar that satisfies

$$Z_5 < 0 \quad \text{and} \quad Z_5 < -\frac{\rho_2}{2a(1)}\left(T_7 - \frac{\rho_1}{\rho_2}T_8 - 2a(1)M_x(1)\right),$$

there exists a scalar $\zeta_3 > 0$ such that

$$T_7 - \frac{\rho_1}{\rho_2}T_8 - 2a(1)M_x(1) + \frac{2a(1)}{\rho_2}Z_5 = -\zeta_3.$$

Thus, for the case when $\rho_1 \neq 0$ and $\rho_2 \neq 0$, there exists a scalar $\zeta_3 > 0$ such that

$$\frac{d}{dt}V(w(\cdot,t)) \leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle - \zeta_3 z(1,t)^2. \tag{6.13}$$

From Equations (6.11)-(6.13) we conclude that there exist scalars $\zeta_1, \zeta_2, \zeta_3 > 0$ such that

$$\frac{d}{dt}V(w(\cdot,t)) \leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle - \zeta z(1,t)^2, \tag{6.14}$$

where $\zeta = \min\{\zeta_1, \zeta_2, \zeta_3\}$.

Since $\zeta < 0$, we conclude that

$$\frac{d}{dt}V(w(\cdot,t)) \leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle.$$

From the theorem hypotheses,

$$\{-T_0 - 2\delta M, -T_1 - 2\delta K_1, -T_2 - 2\delta K_2\} \in \Xi_{d_1,d_2,0}.$$

Thus we conclude that

$$\frac{d}{dt}V(w(\cdot,t)) \leq -2\delta V(w(\cdot,t)), \quad t > 0.$$

Integrating in time yields

$$V(w(\cdot,t)) \leq e^{-2\delta t}V(w(\cdot,0)) \Rightarrow \langle \mathcal{P}z(\cdot,t), z(\cdot,t)\rangle \leq e^{-2\delta t}\langle w_0, \mathcal{P}^{-1}w_0\rangle.$$

Since $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$, $\epsilon\|z(\cdot,t)\|^2 \leq \langle \mathcal{P}z(\cdot,t), z(\cdot,t)\rangle$ and thus

$$\|z(\cdot,t)\| \leq e^{-\delta t}\sqrt{\frac{\langle w_0, \mathcal{P}^{-1}w_0\rangle}{\epsilon}}.$$

Since $z = \mathcal{P}^{-1}w$, $w = \mathcal{P}z$, and therefore,

$$\|w(\cdot,t)\| = \|(\mathcal{P}z)(\cdot,t)\| \leq \|\mathcal{P}\|_{\mathcal{L}}\|z(\cdot,t)\| \leq e^{-\delta t}\|\mathcal{P}\|_{\mathcal{L}}\sqrt{\frac{\langle w_0, \mathcal{P}^{-1}w_0\rangle}{\epsilon}}.$$

Setting

$$M = \|\mathcal{P}\|_{\mathcal{L}}\sqrt{\frac{\langle w_0, \mathcal{P}^{-1}w_0\rangle}{\epsilon}}$$

completes the proof. □

6.1.1 Numerical Results. To illustrate the effectiveness of the controller synthesis, we construct exponentially stabilizing boundary controllers for the following two parabolic PDEs:

$$w_t(x,t) = w_{xx}(x,t) + \lambda w(x,t), \text{ and} \tag{6.15}$$

$$\begin{aligned} w_t(x,t) = &\left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t) \\ &+ \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(x,t), \end{aligned} \tag{6.16}$$

where λ is a scalar which may be chosen freely. We consider the following boundary conditions for these two equations:

$$\text{Dirichlet:} = w(0) = 0, \qquad w(1) = u(t), \tag{6.17}$$

$$\text{Neumann:} = w_x(0) = 0, \qquad w_x(1) = u(t), \tag{6.18}$$

$$\text{Mixed:} = w(0) = 0, \qquad w_x(1) = u(t), \tag{6.19}$$

$$\text{Robin:} = w(0) + w_x(0) = 0, \quad w(1) + w_x(1) = u(t). \tag{6.20}$$

We apply Theorem 6.4 to these PDEs for different degrees of polynomial representation for parameter values $\epsilon = \delta = 0.001$. Table 6.1 and Figure 6.1 illustrate the maximum λ as a function of $d_1 = d_2 = d$ for which we can construct an exponentially

Table 6.1. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.4 are feasible, thereby implying the existence of an exponentially stabilizing controller for Equation (6.15).

Boundary Conditions	$d = 6$	7	8	9	10	11
Dirichlet						
$w(0) = 0, w(1) = u(t)$	$\lambda = 10.3767$	14.3982	17.9626	22.8645	23.3093	27.1179
Neumann						
$w_x(0) = 0, w_x(1) = u(t)$	10.5743	13.1227	16.6992	17.1814	21.8781	21.8781
Mixed						
$w(0) = 0, w_x(1) = u(t)$	10.3767	14.3982	17.9626	22.8645	23.3093	27.1179
Robin						
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	9.3170	12.0911	14.9445	16.6565	18.7748	18.7748

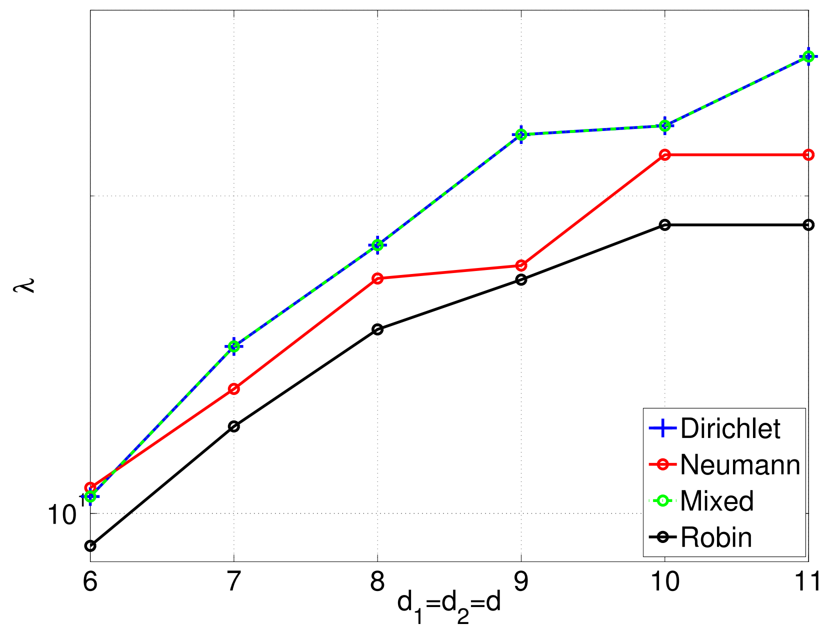


Figure 6.1. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.4 are feasible, thereby implying the existence of an exponentially stabilizing controller for Equation (6.15).

stabilizing controller for Equation (6.15) using the analysis presented in Theorem 6.4.

Similarly Table 6.2 and Figure 6.2 illustrate the maximum λ for which we can construct an exponentially stabilizing controller for Equation (5.17) using the analysis presented in Theorem 6.4.

From Tables 6.1-6.2 we conjecture that if the system is controllable for some suitable definition of controllability, then the conditions of Theorem. 6.4 will be feasible for sufficiently high d_1 and d_2. We emphasize, however, that this is only a conjecture and additional work must be done in order to make this statement rigorous and determine its veracity. A further caveat to these results is the observation that the maximum degree d_1 and d_2 for which the conditions can be tested is a function of the memory and processing speed of the computational platform on which

Table 6.2. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.4 are feasible, thereby implying the existence of an exponentially stabilizing controller for Equation (6.16).

Boundary Conditions	$d = 4$	5	6	7	8
Dirichlet					
$w(0) = 0, w(1) = u(t)$	$\lambda = 19.0216$	36.1359	39.7247	43.5974	44.5219
Neumann					
$w_x(0) = 0, w_x(1) = u(t)$	16.8152	31.3484	32.8186	32.8186	37.5130
Mixed					
$w(0) = 0, w_x(1) = u(t)$	19.0216	36.1359	39.7247	43.5974	44.5219
Robin					
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	12.7869	26.7517	28.0090	28.0090	32.6233

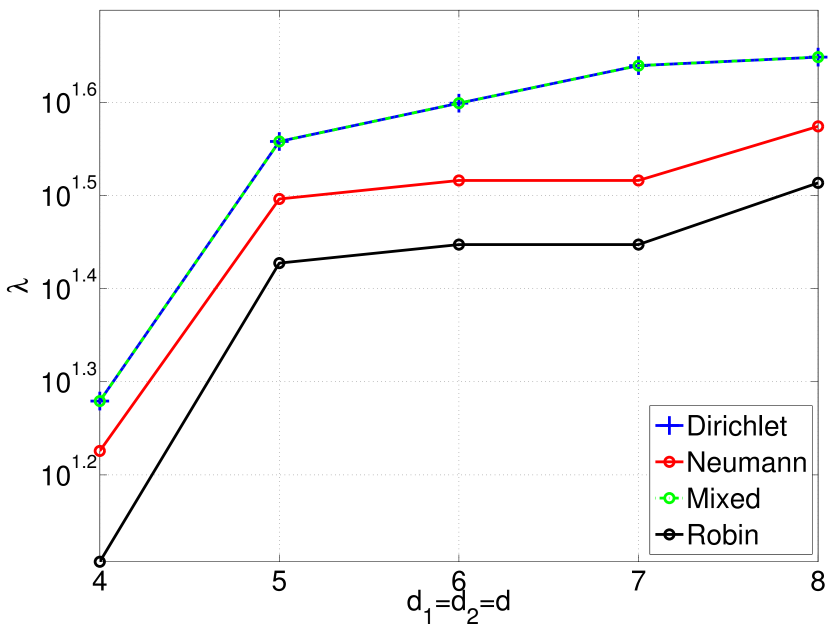


Figure 6.2. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.4 are feasible, thereby implying the existence of an exponentially stabilizing controller for Equation (6.16).

the experiments are performed. Specifically, the number of optimization variables in the underlying SDP problem is determined by the number of polynomial coefficients which scales as $O(d^2)$. To illustrate, all numerical experiments presented in this work were performed on a machine with 8 gigabytes of random access memory, which limited our analysis to a maximum degree of $d_1 = d_2 = 11$ for PDE (6.15) and $d_1 = d_2 = 8$ for PDE (6.16).

Recall that in Theorem 6.4 we use a Lyapunov function of the form $V(w) = \langle w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle$, where, for any $z \in L_2(0,1)$, we define

$$(\mathcal{P}z)(x) = M(x)z(x) + \int_0^x K_1(x,\xi)z(\xi)d\xi + \int_x^1 K_1(x,\xi)z(\xi)d\xi.$$

This form is atypical for the study of PDEs and thus, one may question the necessity of the kernels K_1 and K_2. Especially since their inclusion significantly complicates the analysis. Therefore, to justify the inclusion of the kernels, we test the conditions of

Theorem 6.4 on Equations (6.15)-(6.16) with the constraint $K_1 = K_2 = 0$. Tables 6.3-6.4 present these results.

Table 6.3. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.4 are feasible with $K_1 = K_2 = 0$, thereby implying the existence of an exponentially stabilizing controller for Equation (6.15).

Boundary Conditions	$d = 1$	2	3	$4 \cdots 10$
Dirichlet				
$w(0) = 0, w(1) = u(t)$	$\lambda = 3.90$	4.78	4.88	4.88
Neumann				
$w_x(0) = 0, w_x(1) = u(t)$	3.22	3.51	3.51	3.51
Mixed				
$w(0) = 0, w_x(1) = u(t)$	3.90	4.78	4.88	4.88
Robin				
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	2.32	2.34	2.34	2.34

Table 6.4. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.4 are feasible with $K_1 = K_2 = 0$, thereby implying the existence of an exponentially stabilizing controller for Equation (6.16).

Boundary Conditions	$d = 1$	2	3	$4 \cdots 10$
Dirichlet				
$w(0) = 0, w(1) = u(t)$	$\lambda = 3.51$	7.03	8.59	8.59
Neumann				
$w_x(0) = 0, w_x(1) = u(t)$	3.51	5.46	6.64	6.64
Mixed				
$w(0) = 0, w_x(1) = u(t)$	3.51	7.03	8.59	8.59
Robin				
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	3.51	5.46	5.46	5.46

Comparing Tables 6.1-6.2 with Tables 6.3-6.4 we observe that inclusion of K_1 and K_2 allow us to synthesize controllers for significantly larger values of the parameter $\lambda > 0$. Additionally, it is clear from Tables 6.3-6.4 that with the constraint $K_1 = K_2 = 0$, the maximum value of feasible λ seems to converge. Therefore, we conclude that the kernels K_1 and K_2 play a crucial role in the synthesis of state-feedback controllers.

Finally, we provide a numerical simulation of Equation (6.16) for $\lambda = 20$ and mixed boundary conditions while being acted upon by controllers designed using Theorem 6.4. Figure 6.3 shows the response of the autonomous system ($u(t) = 0$)

with and initial condition

$$e^{-\frac{-(x-0.3)^2}{2(0.07)^2}} - e^{-\frac{-(x-0.7)^2}{2(0.07)^2}}.$$

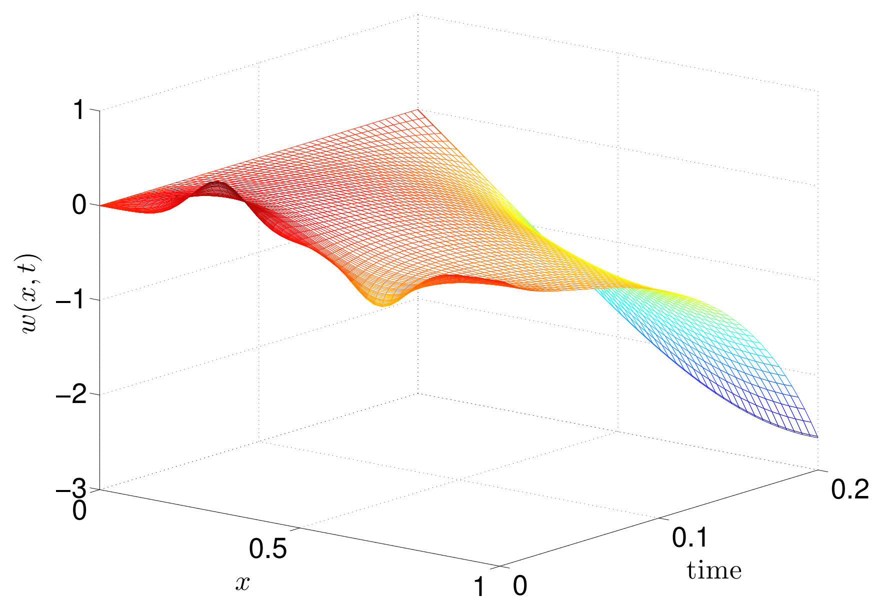


Figure 6.3. Autonomous state evolution of Equation (6.16) for $\lambda = 20$ and mixed boundary conditions .

Figures 6.4-6.5 show the closed loop response of the same PDE and the control effort respectively.

6.2 L_2 Optimal Control

In this section, we consider the inhomogeneous version of Equations (6.1)-(6.2) given by

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t) + f(x,t), \quad x \in [0,1], \quad t \geq 0, \tag{6.21}$$

with boundary conditions of the form

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{and} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t). \tag{6.22}$$

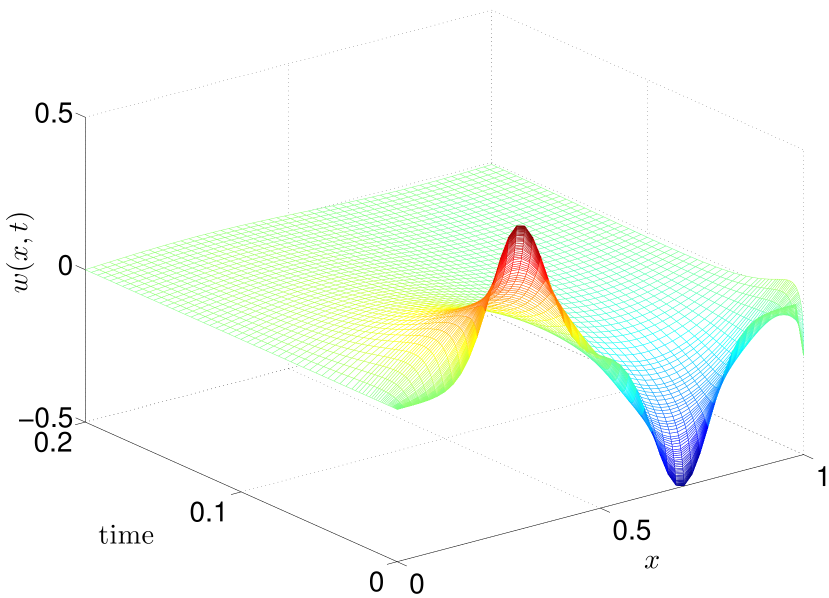


Figure 6.4. Closed loop state evolution of Equation (6.16) for $\lambda = 20$ and mixed boundary conditions .

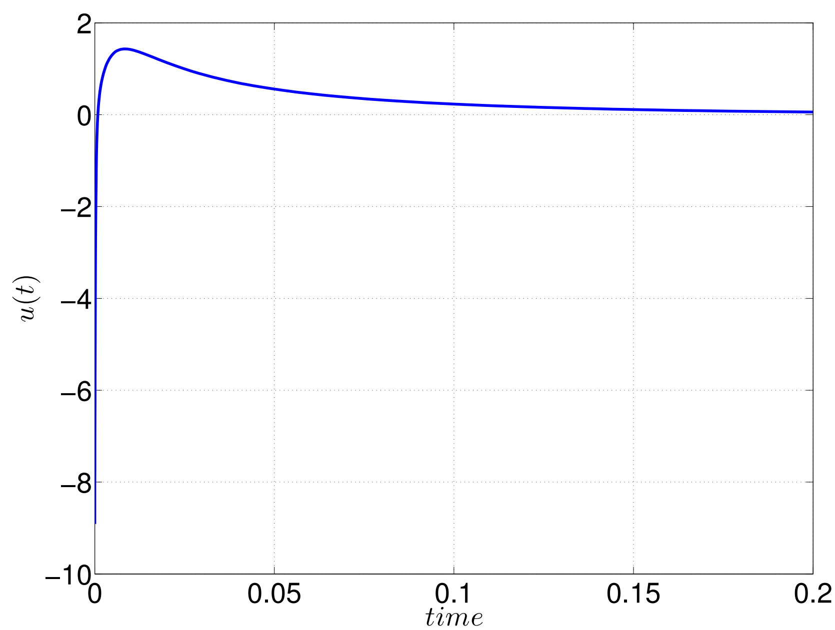


Figure 6.5. Control effort evolution of Equation (6.16) for $\lambda = 20$ and mixed boundary conditions .

Here, the function $f \in C^1_{loc}([0,\infty]; L_2(0,1))$ or $f \in L_2^{loc}([0,\infty]; L_2(0,1))$[2] is the exogenous input. For this system, we wish to synthesize a controller $\mathcal{F} : H^2(0,1) \to \mathbb{R}$ such that if the control input is given by

$$u(t) = \mathcal{F}w(\cdot,t),$$

then there exists a positive scalar γ such that

$$\int_0^\infty \|w(\cdot,t)\|^2 dt \leq \gamma \int_0^\infty \|f(\cdot,t)\|^2 dt.$$

The following assumption, akin to Assumption 6.1, establishes uniqueness and existence of the solutions for the inhomogeneous system.

Assumption 6.5. *For any operator* $\mathcal{F} : H^2(0,1) \to \mathbb{R}$*, initial condition* $w_0 \in \mathcal{D}$ *and* $f \in C^1_{loc}([0,\infty]; L_2(0,1))$*, there exists a classical solution to Equations* (6.21)-(6.22) *with* $u(t) = \mathcal{F}w(\cdot,t)$*. Similarly, for any initial condition* $w_0 \in L_2(0,1)$ *and* $f \in L_2^{loc}([0,\infty]; L_2(0,1))$*, there exists a weak solution to Equations* (6.21)-(6.22)*.*

We present the following theorem for L_2 stability analysis.

Theorem 6.6. *Suppose that there exist scalars* $0 < \epsilon_1 < \epsilon_2$*,* $\gamma > 0$ *and* $\{M, K_1, K_2\} \in \Omega_{d_1,d_2,\epsilon_2,\epsilon_2}$ *such that*

$$\{-T_0 - 2\delta M, -T_1 - 2\delta K_1, -T_2 - 2\delta K_2\} \in \Xi_{d_1,d_2,0},$$

$$T_4(x) = T_5(x) = T_6(x) = 0, \quad T_3 \leq 0,$$

for all m_j*,* $j \in \{1,\cdots,3\}$ *where*

$$\delta = \sqrt{\frac{\epsilon_2}{\epsilon_1\gamma}},$$

m_j *are given by Definition 6.2 and*

$$\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \mathcal{N}(M, K_1, K_2).$$

[2]Refer to the section on notation for definitions of the function spaces.

Define the operator $\mathcal{F} := \mathcal{Z}\mathcal{P}^{-1}$ where, for any $y \in H^2(0,1)$,

$$\mathcal{Z}y = \begin{cases} Z_1 y(1) + \int_0^1 Z_2(x) y(x) dx & \rho_1 = 0, \rho_2 \neq 0 \\ Z_3 y_x(1) + \int_0^1 Z_4(x) y(x) dx & \rho_1 \neq 0, \rho_2 = 0 \\ Z_5 y(1) + \int_0^1 Z_6(x) y(x) dx & \rho_1 \neq 0, \rho_2 \neq 0 \end{cases} \cdot$$

Here, Z_1, Z_3 and Z_5 are any scalars that satisfy

$$\begin{aligned} &Z_1 < 0 \quad \text{and} \quad Z_1 < -\frac{\rho_2}{2a(1)} \left(T_7 - 2a(1)M_x(1)\right), \\ &Z_3 < 0 \quad \text{and} \quad \frac{1}{Z_3} < -\frac{1}{\rho_1 M(1)} \frac{T_7}{T_8}, \\ &Z_5 < 0 \quad \text{and} \quad Z_5 < -\frac{\rho_2}{2a(1)} \left(T_7 - \frac{\rho_1}{\rho_2} T_8 - 2a(1)M_x(1)\right), \end{aligned}$$

and polynomials $Z_2(x)$, $Z_4(x)$ and $Z_6(x)$ are defined as

$$\begin{aligned} Z_2(x) &= \rho_2 K_{1,x}(1,x), \\ Z_4(x) &= \rho_1 K_1(1,x), \\ Z_6(x) &= \rho_2 \left(\frac{\rho_1}{\rho_2} K_1(1,x) + K_{1,x}(1,x)\right). \end{aligned}$$

Additionally,

$$\left(\mathcal{P}y\right)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1).$$

Then any solution w of (6.21) - (6.22) *with $u(t) = (\mathcal{F}w)(t)$ and $w_0 = 0$ satisfies*

$$\int_0^\infty \|w(\cdot,t)\|^2 dt \leq \gamma \int_0^\infty \|f(\cdot,t)\|^2 dt.$$

Proof. Consider the following Lyapunov function

$$V(w(\cdot,t)) = \left\langle w(\cdot,t), \mathcal{P}^{-1} w(\cdot,t)\right\rangle.$$

Taking the time derivative along trajectories of the system, we have

$$\frac{d}{dt} V(w(\cdot,t)) = \left\langle w_t(\cdot,t), \mathcal{P}^{-1} w(\cdot,t)\right\rangle + \left\langle w(\cdot,t), \mathcal{P}^{-1} w_t(\cdot,t)\right\rangle$$

$$= \left\langle \mathcal{A}w(t), \mathcal{P}^{-1}w(t) \right\rangle + \left\langle \mathcal{P}^{-1}w(t), \mathcal{A}w(t) \right\rangle$$
$$+ 2 \left\langle f(\cdot,t), \mathcal{P}^{-1}w(\cdot,t) \right\rangle,$$

where we have used the fact that $\mathcal{P} = \mathcal{P}^\star$ implies $\mathcal{P}^{-1} = (\mathcal{P}^\star)^{-1}$.

Now let $z = \mathcal{P}^{-1}w$. Then

$$\begin{aligned}\frac{d}{dt}V(w(\cdot,t)) &= \left\langle \mathcal{A}\mathcal{P}\mathcal{P}^{-1}w(\cdot,t), \mathcal{P}^{-1}w(\cdot,t) \right\rangle + \left\langle \mathcal{P}^{-1}w(\cdot,t), \mathcal{A}\mathcal{P}\mathcal{P}^{-1}w(\cdot,t) \right\rangle \\ &\quad + 2 \left\langle f(\cdot,t), \mathcal{P}^{-1}w(\cdot,t) \right\rangle \\ &= \left\langle \mathcal{A}\mathcal{P}z(\cdot,t), z(\cdot,t) \right\rangle + \left\langle z(\cdot,t), \mathcal{A}\mathcal{P}z(\cdot,t) \right\rangle + 2 \left\langle f(\cdot,t), z(\cdot,t) \right\rangle.\end{aligned}$$

From the analysis presented in Theorem 6.4, we have

$$\frac{d}{dt}V(w(\cdot,t)) \le \left\langle z(\cdot,t), \mathcal{T}z(\cdot,t) \right\rangle + 2 \left\langle f(\cdot,t), z(\cdot,t) \right\rangle.$$

Thus,

$$\begin{aligned}&\frac{d}{dt}V(w(\cdot,t)) + \delta \left\langle z(\cdot,t), \mathcal{P}z(\cdot,t) \right\rangle - \frac{1}{\delta} \left\langle f(\cdot,t), \mathcal{P}^{-1}f(\cdot,t) \right\rangle \\ &\le \left\langle z(\cdot,t), (\mathcal{T} + \delta\mathcal{P})z(\cdot,t) \right\rangle + 2 \left\langle f(\cdot,t), z(\cdot,t) \right\rangle - \frac{1}{\delta} \left\langle f(\cdot,t), \mathcal{P}^{-1}f(\cdot,t) \right\rangle \\ &= \left\langle \begin{bmatrix} z(\cdot,t) \\ f(\cdot,t) \end{bmatrix}, \begin{bmatrix} \mathcal{T} + \delta\mathcal{P} & \mathcal{I} \\ \mathcal{I} & -\frac{1}{\delta}\mathcal{P}^{-1} \end{bmatrix} \begin{bmatrix} z(\cdot,t) \\ f(\cdot,t) \end{bmatrix} \right\rangle.\end{aligned} \tag{6.23}$$

From Schur complement, the operator

$$\begin{bmatrix} \mathcal{T} + \delta\mathcal{P} & \mathcal{I} \\ \mathcal{I} & -\frac{1}{\delta}\mathcal{P}^{-1} \end{bmatrix} \le 0$$

if and only if

$$\mathcal{T} + 2\delta\mathcal{P} \le 0.$$

Since $\{-T_0 - 2\delta M, -T_1 - 2\delta K_1, -T_2 - 2\delta K_2\} \in \Xi_{d_1,d_2,0}$, we have that

$$\mathcal{T} + 2\delta\mathcal{P} \le 0,$$

and consequently, from Equation (6.23),

$$\frac{d}{dt}V(w(\cdot,t)) + \delta \langle z(\cdot,t), \mathcal{P}z(\cdot,t)\rangle \leq \frac{1}{\delta}\left\langle f(\cdot,t), \mathcal{P}^{-1}f(\cdot,t)\right\rangle.$$

Integrating in time from $t = 0$ to $t = T < \infty$, we obtain

$$\begin{aligned}&V(w(\cdot,T)) - V(w(\cdot,0)) + \delta \int_0^T \langle z(\cdot,t), \mathcal{P}z(\cdot,t)\rangle\, dt\\&\leq \frac{1}{\delta}\int_0^T \left\langle f(\cdot,t), \mathcal{P}^{-1}f(\cdot,t)\right\rangle dt.\end{aligned}$$

Since $w_0(x) = w(x,0) = 0$, $V(w(\cdot,0)) = 0$. Additionally, $V(w(\cdot,T)) \geq 0$, thus

$$\int_0^T \langle z(\cdot,t), \mathcal{P}z(\cdot,t)\rangle\, dt \leq \frac{1}{\delta^2}\int_0^T \left\langle f(\cdot,t), \mathcal{P}^{-1}f(\cdot,t)\right\rangle dt.$$

Since, $\langle z(\cdot,t), \mathcal{P}z(\cdot,t)\rangle = \langle w(\cdot,t), \mathcal{P}^{-1}w(\cdot,t)\rangle$,

$$\int_0^T \left\langle w(\cdot,t), \mathcal{P}^{-1}w(\cdot,t)\right\rangle dt \leq \frac{1}{\delta^2}\int_0^T \left\langle f(\cdot,t), \mathcal{P}^{-1}f(\cdot,t)\right\rangle dt.$$

Since $\{M, K_1, K_2\} \in \Omega_{d_1,d_2,\epsilon_1,\epsilon_2}$, we have from Lemma C.1 that

$$\begin{aligned}&\frac{1}{\epsilon_2}\|w(\cdot,t)\|^2 \leq \left\langle w(\cdot,t), \mathcal{P}^{-1}w(\cdot,t)\right\rangle \quad \text{and}\\&\left\langle f(\cdot,t), \mathcal{P}^{-1}f(\cdot,t)\right\rangle \leq \frac{1}{\epsilon_1}\|f(\cdot,t)\|^2.\end{aligned}$$

Therefore,

$$\frac{1}{\epsilon_2}\int_0^T \|w(\cdot,t)\|^2 dt \leq \frac{1}{\epsilon_1\delta^2}\int_0^T \|f(\cdot,t)\|^2 dt.$$

Consequently,

$$\int_0^T \|w(\cdot,t)\|^2 dt \leq \frac{\epsilon_2}{\epsilon_1\delta^2}\int_0^T \|f(\cdot,t)\|^2 dt.$$

Since

$$\delta = \sqrt{\frac{\epsilon_2}{\epsilon_1\gamma}},$$

we obtain

$$\int_0^T \|w(\cdot,t)\|^2 dt \leq \gamma\int_0^T \|f(\cdot,t)\|^2 dt.$$

Taking the limit $T \to \infty$ completes the proof. □

6.2.1 Numerical Results.

We now test the conditions of Theorem 6.6 on the perturbed versions of Equations (6.15)-(6.16), namely

$$w_t(x,t) = w_{xx}(x,t) + \lambda w(x,t) + f(x,t), \text{ and} \tag{6.24}$$

$$w_t(x,t) = \left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t) \\ + \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(x,t) + f(x,t), \tag{6.25}$$

where $f \in L_2(0,\infty; L_2(0,1))$ is the exogenous distributed input. We consider the following boundary conditions for these two equations:

$$\text{Dirichlet:} = w(0) = 0, \qquad w(1) = u(t), \tag{6.26}$$

$$\text{Neumann:} = w_x(0) = 0, \qquad w_x(1) = u(t), \tag{6.27}$$

$$\text{Mixed:} = w(0) = 0, \qquad w_x(1) = u(t), \tag{6.28}$$

$$\text{Robin:} = w(0) + w_x(0) = 0, \quad w(1) + w_x(1) = u(t). \tag{6.29}$$

Additionally, we choose the values for the parameter λ so that the autonomous unperturbed PDEs are unstable. The chosen values of λ for each case are presented in Table 6.5.

Table 6.5. Values of parameter λ chosen for Equations (6.24)-(6.25) with boundary conditions (6.26)-(6.29).

	Dirichlet	Neumann	Mixed	Robin
PDE (6.24)	$\lambda = \pi^2 + 0.04$	0.033	$\frac{\pi^2}{4} + 0.034$	-0.967
PDE (6.25)	$\lambda = 19.006$	-0.195	4.72	-2.37

We apply Theorem 6.6 to these PDEs for different degrees of polynomial representation for parameter values $\epsilon_1 = 0.001$ and $\epsilon_2 = 1$, and find the smallest upper bound of the state $\gamma > 0$. Table 6.6 and Figure 6.6 illustrate the minimum γ as a function of $d_1 = d_2 = d$ for which we can construct an optimal controller for Equation (6.24) using the analysis presented in Theorem 6.6.

Table 6.6. Minimum γ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.6 are feasible, thereby implying the existence of an optimal controller for Equation (6.24).

Boundary Conditions	$d = 3$	4	5	6	7
Dirichlet					
$w(0) = 0, w(1) = u(t)$	$\gamma = 99.90$	99.90	99.90	91.79	27.73
Neumann					
$w_x(0) = 0, w_x(1) = u(t)$	220.58	45.89	17.08	10.74	10.74
Mixed					
$w(0) = 0, w_x(1) = u(t)$	999.93	176.65	32.71	7.515	5.615
Robin					
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	125	39.06	19.04	13.18	13.18

Table 6.7 and Figure 6.7 illustrate the minimum γ as a function of $d_1 = d_2 = d$ for which we can construct an optimal controller for Equation (6.25) using the analysis presented in Theorem 6.6.

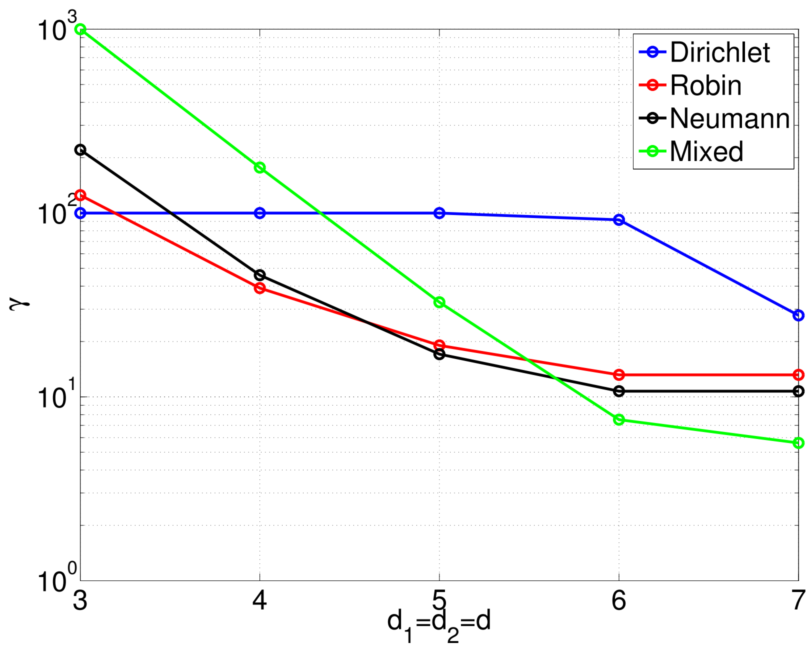


Figure 6.6. Minimum γ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.6 are feasible, thereby implying the existence of an optimal controller for Equation (6.24).

Table 6.7. Minimum γ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.6 are feasible, thereby implying the existence of an optimal controller for Equation (6.25).

Boundary Conditions	$d = 3$	4	5	6	7
Dirichlet					
$w(0) = 0, w(1) = u(t)$	$\gamma = 99.90$	99.90	7.76	7.03	7.03
Neumann					
$w_x(0) = 0, w_x(1) = u(t)$	343.73	3.78	2.92	2.29	2.29
Mixed					
$w(0) = 0, w_x(1) = u(t)$	999.93	4.88	1.71	1.46	1.46
Robin					
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	134.70	3.84	3.66	3.41	3.41

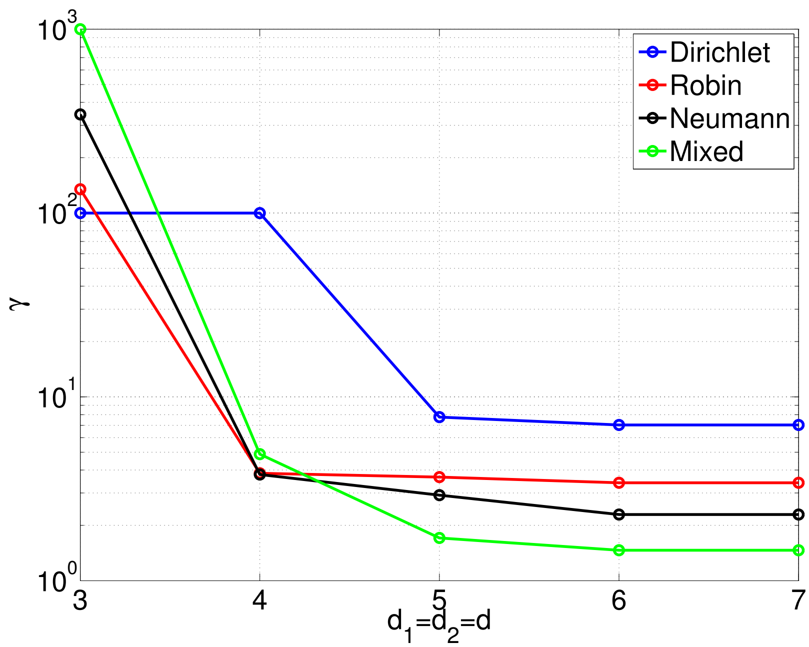


Figure 6.7. Minimum γ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 6.6 are feasible, thereby implying the existence of an optimal controller for Equation (6.25).

As for the case of exponentially stabilizing control, these results suggest that increasing the degree of polynomial representation leads to the construction of an controller for a lower value of $\gamma > 0$. However, for optimal control synthesis this effect is not as pronounced as for the case of exponentially stabilizing control. Moreover, for Equation (6.25) the values of γ seem to converge. We would require to test the conditions of Theorem 6.6 for higher degrees of polynomial representation. However, as discussed previously, that would incur a penalty on the memory requirements of the machine on which these tests are performed.

Since the conditions of Theorems 6.4 and 6.6 are similar, we infer that setting $K_1 = K_2 = 0$ would worsen the performance of the controllers synthesized.

Finally, we present a numerical simulation for the optimal controller. We simulate Equation (6.25) with exogenous input chosen as

$$f(x,t) = e^{-0.1t}\cos\left(\frac{\pi t}{5}\right)(1+\sin(0.1\pi x)).$$

Figure 6.8 shows the state evolution, from a zero initial condition, without any control input.

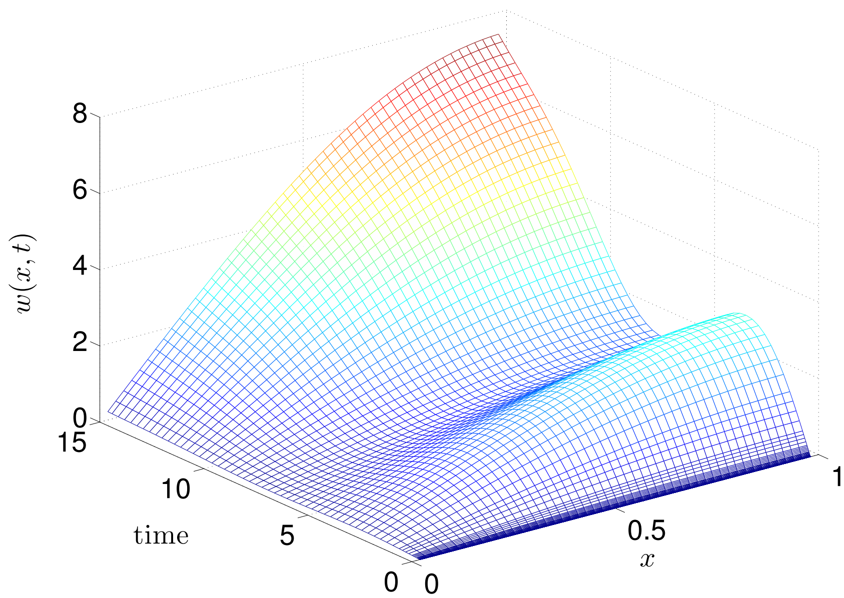


Figure 6.8. Autonomous state evolution of Equation (6.25) with mixed boundary conditions .

Figures 6.9-6.10 show the closed loop response of the same PDE and the control effort respectively.

6.3 Inverses of Positive Operators

In Theorems 6.4 and 6.6 we construct operators $\mathcal{Z}$ and $\mathcal{P}$ satisfying the conditions of the respective theorems. If such operators exist, then the controller is given by $\mathcal{F} = \mathcal{Z}\mathcal{P}^{-1}$. Thus, given a positive operator $\mathcal{P}$, we require a method of constructing $\mathcal{P}^{-1}$. Therefore, in this section, given scalar valued polynomials $\{M, K_1, K_2\} \in \Xi_{\{d_1,d_2,\epsilon\}}$, or indeed $\{M, K_1, K_2\} \in \Omega_{\{d_1,d_2,\epsilon_1,\epsilon_2\}}$ for any $0 < \epsilon_1 < \epsilon_2$, we

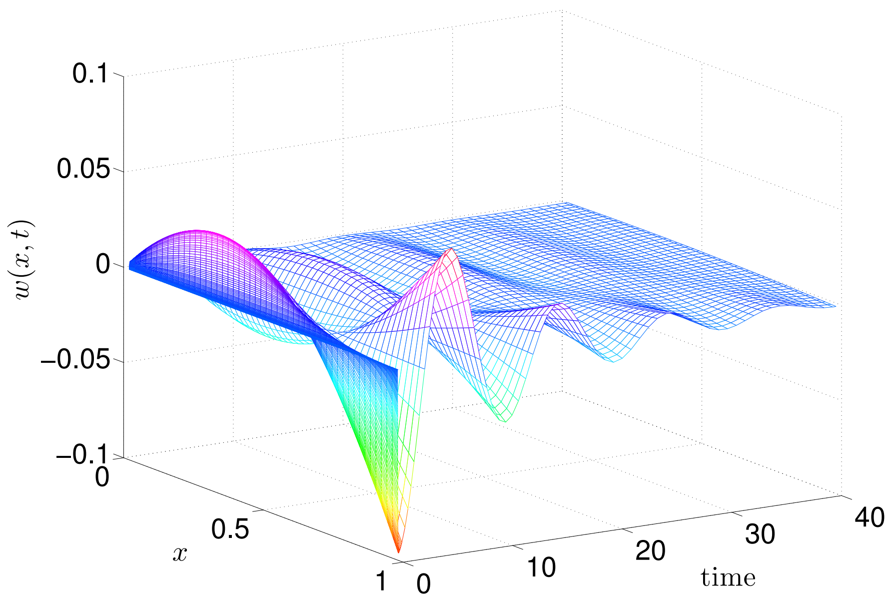


Figure 6.9. Closed loop state evolution of Equation (6.25) with mixed boundary conditions .

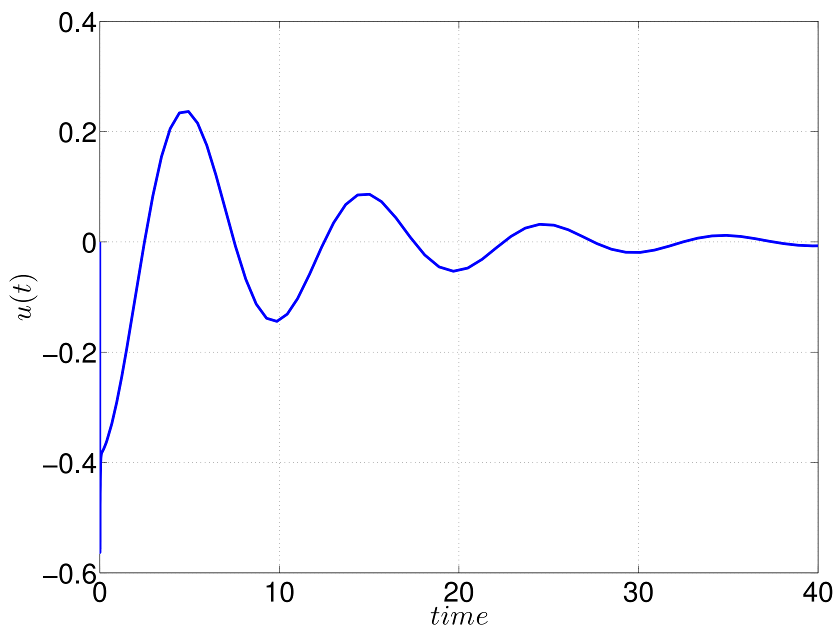


Figure 6.10. Control effort evolution of Equation (6.25) with mixed boundary conditions .

will provide a method to construct $\mathcal{P}^{-1}$ where

$$\left(\mathcal{P}y\right)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi.$$

For operators without joint positivity, this procedure has been presented in [82] and expanded in [83]. In this section, we further expand these results by proposing a method for constructing inverses for the class of operators considered in Section 5.2.

Since all positive bounded linear operators are invertible [35], the operators constructed in Theorem 5.5 are invertible. Of course, to construct the inverses of such operators, one could enforce the supremum of the integral kernels $K_i(x,\xi)$, $i \in \{1,2\}$ to be less than the infimum of $M(x)$ so that the power series expansion of the inverse operator converges. However, such conditions are very conservative. Our approach uses the results presented in [84] where it has been shown that operators belonging to the set $\Xi_{\{d_1,d_2,\epsilon\}}$ are the input-output maps of well-posed Linear Time Varying (LTV) systems. Thus, by switching the input and the output, such operators can be inverted. We prove this fact explicitly.

Let $\{M, K_1, K_2\} \in \Xi_{\{d_1,d_2,\epsilon\}}$, then $K_1(x,\xi)$ and $K_2(x,\xi)$ are of degree d_2+1 in variables x and ξ. We can always find a matrix $R \in \mathbb{R}^{d_2+2\times d_2+2}$ such that $K_1(x,\xi) = Z_{d_2+1}(x)^T R Z_{d_2+1}(\xi)$. Recall that we denote the vector of monomials up to degree d_2+1 by $Z_{d_2+1}(\cdot)$. Since, $K_2(x,\xi) = K_1(x,\xi)$, we get $K_2(x,\xi) = Z_{d_2+1}(x)^T R^T Z_{d_2+1}(\xi)$. Let $R = R_1 R_2$ be a factorization, for e.g. QR factorization, then

$$\begin{aligned}
K_1(x,\xi) &= Z_{d_2+1}(x)^T R_1 R_2 Z_{d_2+1}(\xi),\\
K_2(x,\xi) &= Z_{d_2+1}(x)^T R_2^T R_1^T Z_{d_2+1}(\xi).
\end{aligned}$$

With this, we provide the following definition.

Definition 6.7. *Consider the operator*

$$\left(\mathcal{P}y\right)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi,$$

where

$$\{M, K_1, K_2\} \in \Xi_{\{d_1,d_2,\epsilon\}}, \qquad K_1(x,\xi) = Z_{d_2+1}(x)^T R_1 R_2 Z_{d_2+1}(\xi),$$
$$K_2(x,\xi) = Z_{d_2+1}(x)^T R_2^T R_1^T Z_{d_2+1}(\xi), \quad R = R_1 R_2.$$

We define

$$\Theta_{\mathcal{P}} = \{M, F_1, F_2, G_1, G_2\},$$

where

$$\begin{aligned} F_1(x) &= Z_{d_2+1}(x)^T R_1 \in \mathbb{R}^{1\times d_2+1}, \\ F_2(x) &= -Z_{d_2+1}(x)^T R_2^T \in \mathbb{R}^{1\times d_2+1}, \\ G_1(\xi) &= R_2 Z_{d_2+1}(\xi) \in \mathbb{R}^{d_2+1\times 1}, \\ G_2(\xi) &= R_1^T Z_{d_2+1}(\xi) \in \mathbb{R}^{d_2+1\times 1}. \end{aligned}$$

With this definition, if

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi,$$

then $\Theta_{\mathcal{P}} = \{M, F_1, F_2, G_1, G_2\}$ *implies that*

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x F_1(x)G_1(\xi)y(\xi)d\xi - \int_x^1 F_2(x)G_2(\xi)y(\xi)d\xi.$$

We provide the following Lemma which we will use to construct inverse operators.

Lemma 6.8. *Let* $A(x)$ *be a matrix in* $\mathbb{R}^{k\times k}$, $k \in \mathbb{N}$, *whose entries are Lebesgue integrable and continuous on* $x \in [0,1]$. *Then, the matrix differential equation*

$$\begin{aligned} \frac{dU(x)}{dx} =& A(x)U(x), \\ U(0) =& I, \end{aligned}$$

has a unique absolutely continuous solution which is given by the uniform limit on $0 \le x \le 1$ *of the sequence* $U_1(x), U_2(x), \cdots,$ *which are defined recursively as*

$$U_{n+1}(x) = I + \int_0^x A(\xi)U_n(\xi)d\xi, \quad U_1(x) = I.$$

Additionally, $U(x)$ *is non-singular.*

The matrix $U(x)$ *is known as the* ***fundamental matrix of*** $\mathbf{A}(\mathbf{x})$.

A proof is provided in Appendix C. Additionally, refer to [84] and [85] and references therein for a similar proof.

Theorem 6.9. *For* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$, *let*

$$(\mathcal{P}w)(x) = M(x)w(x) + \int_0^t K_1(x,\xi)w(\xi)d\xi + \int_x^1 K_2(x,\xi)w(\xi)d\xi, \quad w \in L_2(0,1).$$

Additionally, let $\Theta_{\mathcal{P}} = (M, F_1, F_2, G_1, G_2)$. *Define the operator* $\hat{\mathcal{P}}$ *as*

$$\left(\hat{\mathcal{P}}w\right)(x) = M(x)^{-1}w(x) - \int_0^x \gamma_1(x,\xi)w(\xi)d\xi - \int_x^1 \gamma_2(x,\xi)w(\xi)d\xi,$$

where

$$\begin{aligned}
\gamma_1(x,\xi) =& M(x)^{-1}C(x)U(x)(I_{4(d+1)} - P)U(\xi)^{-1}B(\xi)M(\xi)^{-1}, \\
\gamma_2(x,\xi) =& -M(x)^{-1}C(x)U(x)PU(\xi)^{-1}B(\xi)M(\xi)^{-1}, \\
B(x) =& \begin{bmatrix} G_1(x) \\ \\ G_2(x) \end{bmatrix}, \quad C(x) = \begin{bmatrix} F_1(x) & F_2(x) \end{bmatrix}, \\
P =& \left(N_1 + N_2U(1)\right)^{-1} N_2U(1), \\
N_1 =& \begin{bmatrix} I_{2(d+1)} & 0 \\ \\ 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 \\ \\ 0 & I_{2(d+1)} \end{bmatrix}, \quad N_1, N_2 \in \mathbb{S}^{4(d+1)}, \\
U(x) =& \textit{fundamental matrix of } -B(x)M(x)^{-1}C(x), \textit{ and}
\end{aligned}$$

$$d = d_2 + 1.$$

Then, $\hat{\mathcal{P}}$ is the inverse of $\mathcal{P}$, i.e. $\mathcal{P}\hat{\mathcal{P}} = \hat{\mathcal{P}}\mathcal{P} = \mathcal{I}$, where $\mathcal{I}$ is the identity operator.

The same result holds for $\{M, K_1, K_2\} \in \Omega_{d_1,d_2,\epsilon_1,\epsilon_2}$ for any $0 < \epsilon_1 < \epsilon_2$.

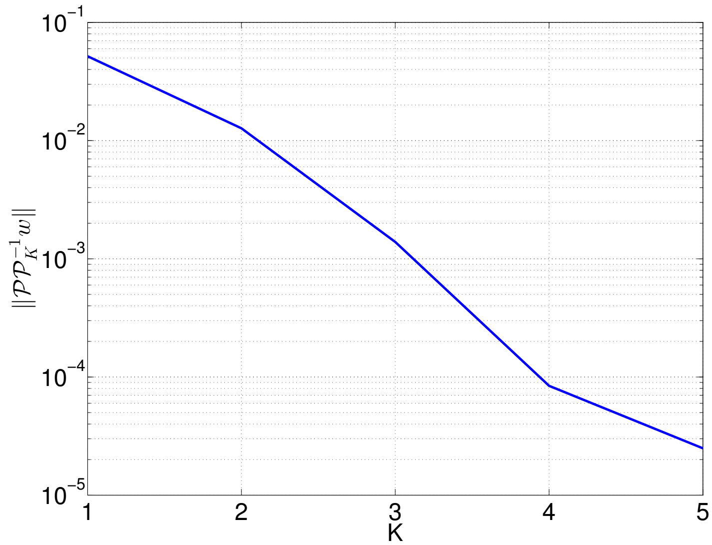
Refer to Appendix C for the proof.

To construct the inverse in practice, the fundamental matrix $U(x)$ has to be replaced by

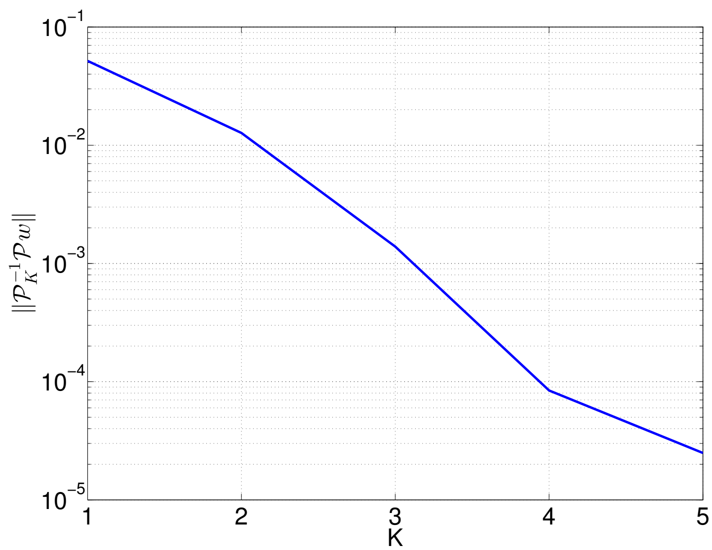
$$U_K(x) = I + \int_0^x \big(- B(\xi)M(\xi)^{-1}C(\xi)\big)U_{K-1}(\xi)d\xi, \quad U_1(x) = I_{4(d+1)},$$

for some finite K where K is chosen sufficiently large so that the inverse is approximated adequately. In practice, we have found that only a few terms are required for convergence. To illustrate, in Figures 6.11(a) and 6.11(b) we find some $(M, K_1, M_2) \in \Omega_{1,1,1,1}$. Then we plot $\|w - \mathcal{P}\mathcal{P}_K^{-1}w\|$ and $\|w - \mathcal{P}_K^{-1}\mathcal{P}w\|$, where $\mathcal{P}_K^{-1}$ denotes $\mathcal{P}^{-1}$ with $U(x)$ replaced by $U_K(x)$, as a function of K for the arbitrarily chosen function $w(x) = x(x - 0.4)(x - 1)$. In this case, $K = 5$ yields norm error of order $\approx 10^{-5}$.

Finally, Figures 6.12(a) and 6.12(b) illustrate $w(t)$, $\left(\mathcal{P}\mathcal{P}_K^{-1}w\right)(t)$ and $\left(\mathcal{P}_K^{-1}\mathcal{P}w\right)(t)$.

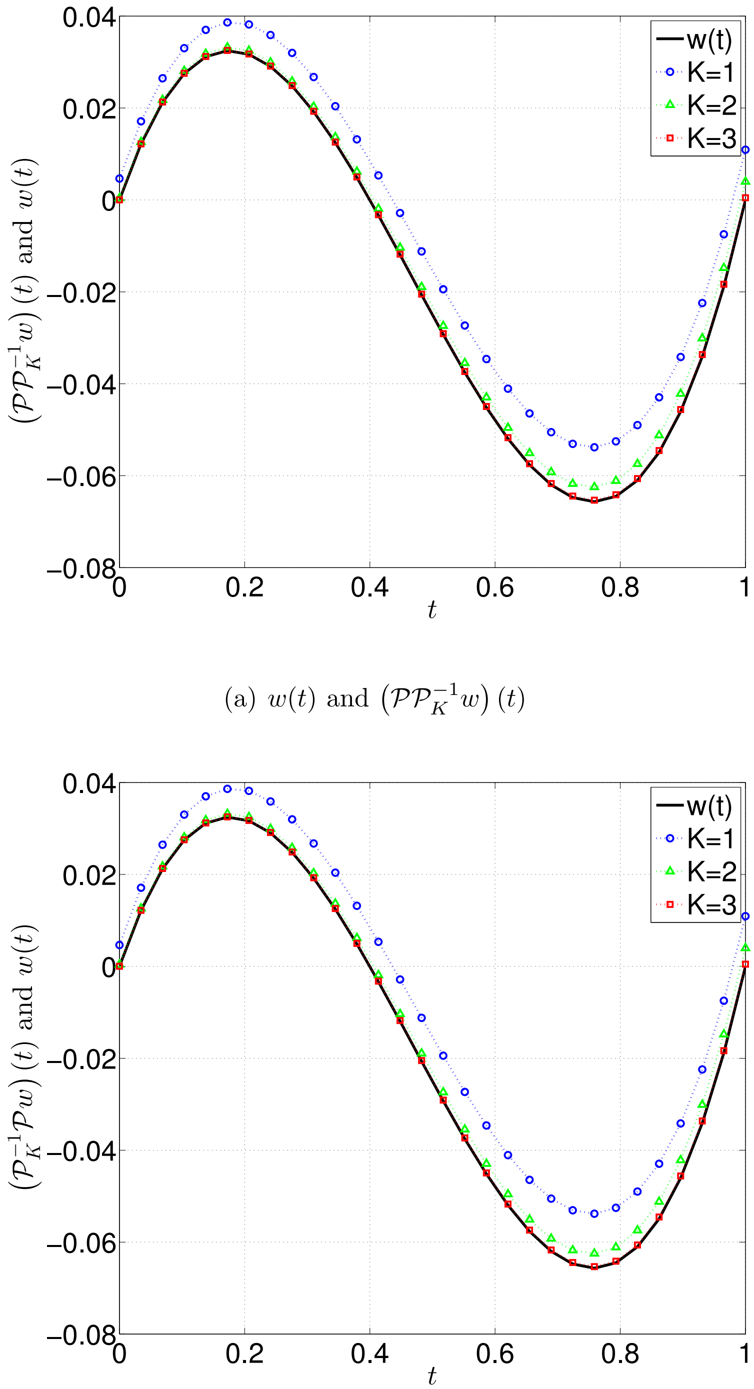


(a) $\|w - \mathcal{P}\mathcal{P}_K^{-1}w\|$



(b) $\|w - \mathcal{P}_K^{-1}\mathcal{P}w\|$

Figure 6.11. $\|w - \mathcal{P}\mathcal{P}_K^{-1}w\|$ and $\|w - \mathcal{P}_K^{-1}\mathcal{P}w\|$ as a function of K.



(a) $w(t)$ and $\left(\mathcal{P}\mathcal{P}_K^{-1}w\right)(t)$

(b) $w(t)$ and $\left(\mathcal{P}_K^{-1}\mathcal{P}w\right)(t)$

Figure 6.12. $w(t)$, $\left(\mathcal{P}\mathcal{P}_K^{-1}w\right)(t)$ and $\left(\mathcal{P}_K^{-1}\mathcal{P}w\right)(t)$ as a function of K.

CHAPTER 7

OBSERVER BASED BOUNDARY CONTROL OF PARABOLIC PDES USING POINT OBSERVATION

In this chapter we consider boundary stabilization of parabolic PDEs when only a partial knowledge of the state is available. In Chapter 6 we considered controller design using the complete knowledge of the state. However, due to the infinite-dimensional nature of PDEs, real-time measurement of the complete state is not possible. Thus, a realistic approach would entail the design of controllers using only the partial knowledge of the state.

We consider Equations (6.1)-(6.2) given by

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t), \quad x \in [0,1], \quad t \geq 0, \tag{7.1}$$

with boundary conditions

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0, \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t), \tag{7.2}$$

and *measurement*

$$y(t) = \mu_1 w(1,t) + \mu_2 w_x(1,t). \tag{7.3}$$

As in Chapter 6, the function $u(t) \in \mathbb{R}$ is the *control input*. The measurement $y(t) \in \mathbb{R}$ is also called an *output*. As in previous chapters, the functions a, b and c are polynomials in x and

$$a(x) \geq \alpha > 0, \quad \text{for} \quad x \in [0,1]. \tag{7.4}$$

The scalars $\nu_i, \rho_j \in \mathbb{R}$, $i,j \in \{1,2\}$, satisfy

$$|\nu_1| + |\nu_2| > 0, \quad \text{and} \quad |\rho_1| + |\rho_2| > 0. \tag{7.5}$$

Additionally, the scalars μ_k, $k \in \{1,2\}$ satisfy

$$\mu_1 \neq 0 \text{ and } \mu_2 = 0 \quad \text{if} \quad \rho_1 = 0$$

$$\mu_1 = 0 \text{ and } \mu_2 \neq 0 \quad \text{if} \quad \rho_2 = 0 \tag{7.6}$$

$$\mu_1 \neq 0 \text{ and } \mu_2 = 0 \quad \text{if} \quad \rho_1 \neq 0 \text{ and } \rho_2 \neq 0. \tag{7.7}$$

The method we use is to design an observer with measurement $y(t)$ as inputs such that the state of the observer estimates the state of the system represented by Equations (7.1)-(7.2). Additionally, the output of the observer is constructed such that if it is set as the input $u(t)$, then the System (7.1)-(7.2) is stabilized. The simplest class of observers for which it is possible to verify closed loop stability is *Luenberger observers*. In our version of the Luenberger observer, the dynamics of the state estimate $\hat{w}$ are defined as

$$\hat{w}_t(x,t) = a(x)\hat{w}_{xx}(x,t) + b(x)\hat{w}_x(x,t) + c(x)\hat{w}(x,t) + p(x,t), \tag{7.8}$$

with boundary conditions

$$\nu_1\hat{w}(0,t) + \nu_2\hat{w}_x(0,t) = 0, \quad \rho_1\hat{w}(1,t) + \rho_2\hat{w}_x(1,t) = q(t) + u(t), \tag{7.9}$$

where $p(x,t)$ and $q(t)$ are the inputs to the observer.

We wish to design a controller $\mathcal{F} : H^2(0,1) \to \mathbb{R}$, observer operator $\mathcal{O} : \mathbb{R} \to L_2(0,1)$, and scalars O such that if the observer is given by Equations (7.8)-(7.9) with the observer inputs given by

$$\begin{aligned} p(x,t) =& \left(\mathcal{O}\left(\hat{y}(t) - y(t)\right)\right)(x), \\ q(t) =& O\left(\hat{y}(t) - y(t)\right), \end{aligned}$$

and the control input is given by

$$u(t) = \mathcal{F}\hat{w}(\cdot,t),$$

then the system represented by Equations (7.1)-(7.2) is stable. Here,

$$\hat{y}(t) = \mu_1\hat{w}(1,t) + \mu_2\hat{w}_x(1,t).$$

With the control input $u(t) = \mathcal{F}\hat{w}(\cdot, t)$, the coupled dynamics of the system state w and the observer state $\hat{w}$ can be written as

$$\begin{aligned} w_t(x,t) =& a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t) \\ \hat{w}_t(x,t) =& a(x)\hat{w}_{xx}(x,t) + b(x)\hat{w}_x(x,t) + c(x)\hat{w}(x,t) + \left(\mathcal{O}\left(\hat{y}(t) - y(t)\right)\right)(x), \end{aligned} \tag{7.10}$$

with boundary conditions

$$\begin{aligned} &\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0, \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = \mathcal{F}\hat{w}(\cdot,t), \\ &\nu_1 \hat{w}(0,t) + \nu_2 \hat{w}_x(0,t) = 0, \quad \rho_1 \hat{w}(1,t) + \rho_2 \hat{w}_x(1,t) = O\left(\hat{y}(t) - y(t)\right) + \mathcal{F}\hat{w}(\cdot,t), \end{aligned} \tag{7.11}$$

where

$$y(t) = \mu_1 w(1,t) + \mu_2 w_x(1,t), \quad \hat{y}(t) = \mu_1 \hat{w}(1,t) + \mu_2 \hat{w}_x(1,t),$$

A block-diagram of the coupled dynamics can be found in Figure 7.1.

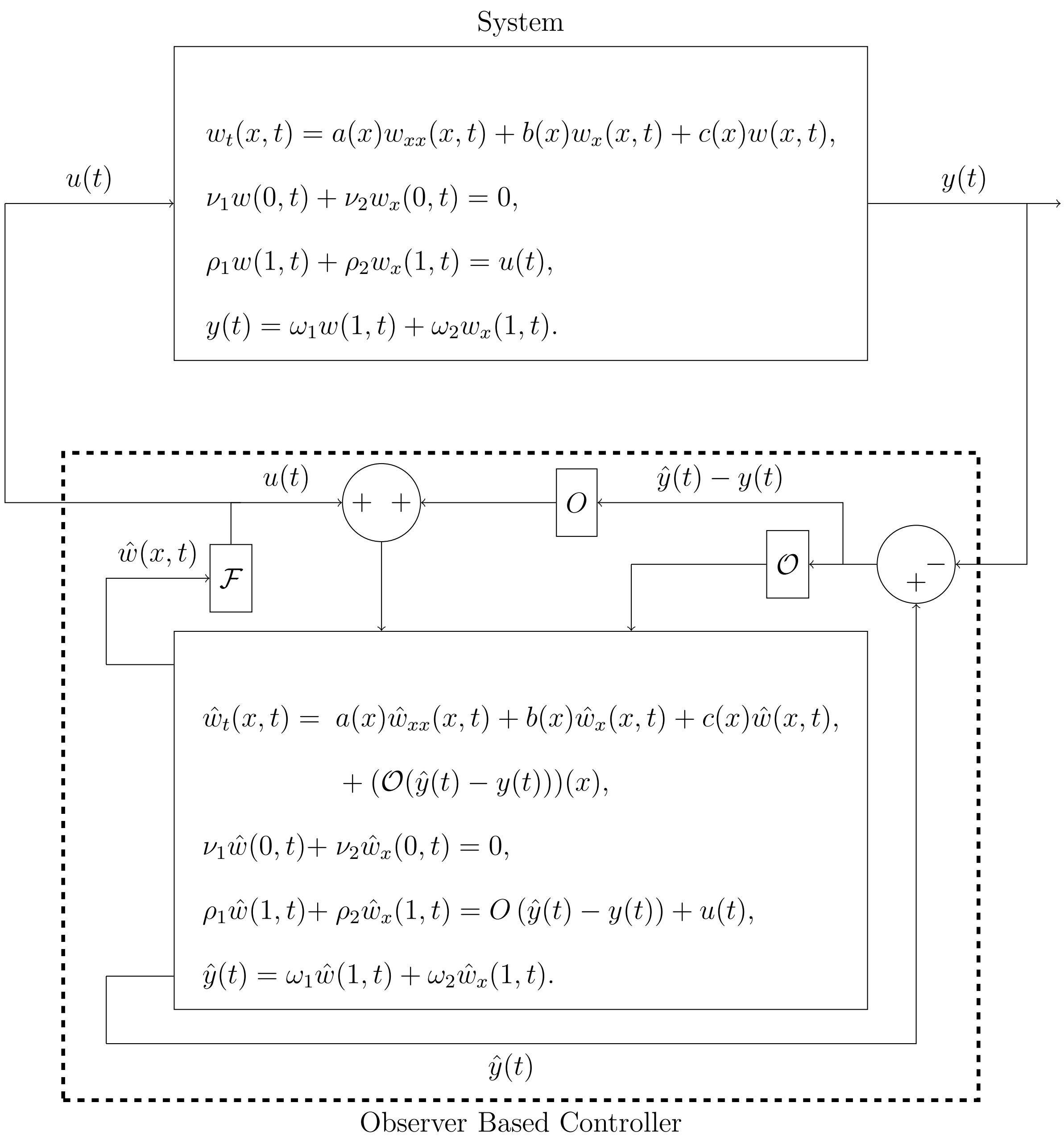


Figure 7.1. Diagram representing the coupled dynamics (7.10)-(7.11)

For the coupled PDEs in the form of Equations (7.10)-(7.11), we define the following first order form

$$\begin{bmatrix} \dot{\mathbf{w}}(t) \\ \dot{\hat{\mathbf{w}}}(t) \end{bmatrix} = \begin{bmatrix} \mathcal{A} & 0 \\ -\mathcal{O}\mathcal{C} & \mathcal{A} + \mathcal{O}\mathcal{C} \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \hat{\mathbf{w}}(t) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{w} \\ \hat{\mathbf{w}} \end{bmatrix} \in \hat{\mathcal{D}},$$

where the operator $\mathcal{A} : H^2(0,1) \to L_2(0,1)$ is defined as

$$(\mathcal{A}z)(x) = a(x)z_{xx}(x) + b(x)z_x(x) + c(x)z(x), \tag{7.12}$$

the operator $\mathcal{C} : H^2(0,1) \to \mathbb{R}$ is defined as

$$\mathcal{C}z = \mu_1 z(1) + \mu_2 z_x(1),$$

and the space $\hat{\mathcal{D}}$ is defined as

$$\hat{\mathcal{D}} = \left\{ \begin{bmatrix} z \\ \hat{z} \end{bmatrix} \in H^2(0,1) \oplus H^2(0,1) : \quad \nu_1 \begin{bmatrix} z(0) \\ \hat{z}(0) \end{bmatrix} + \nu_2 \begin{bmatrix} z_x(0) \\ \hat{z}_x(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and} \right.$$
$$\left. \rho_1 \begin{bmatrix} z(1) \\ \hat{z}(1) \end{bmatrix} + \rho_2 \begin{bmatrix} z_x(1) \\ \hat{z}_x(1) \end{bmatrix} = \begin{bmatrix} 0 & \mathcal{F} \\ -O\mathcal{C} & \mathcal{F} + O\mathcal{C} \end{bmatrix} \begin{bmatrix} z \\ \hat{z} \end{bmatrix} \right\}. \tag{7.13}$$

Similar to Chapter 6, we make the following assumption for the uniqueness and existence of solutions for the coupled closed loop system.

Assumption 7.1. *For any controller $\mathcal{F} : H^2(0,1) \to \mathbb{R}$, observer operator $\mathcal{O} : L_2(0,1) \to L_2(0,1)$, scalar O, and initial condition $\begin{bmatrix} w_0 \\ \hat{w}_0 \end{bmatrix} \in \hat{\mathcal{D}}$, there exists a classical solution to Equations* (7.10)-(7.11) *with control input $u(t) = \mathcal{F}\hat{w}(\cdot,t)$ and*

$$p(x,t) = \left(\mathcal{O}\left(\hat{y}(t) - y(t)\right)\right)(x),$$
$$q(t) = O\left(\hat{y}(t) - y(t)\right).$$

Similarly, for any initial condition $\begin{bmatrix} w_0 \\ \hat{w}_0 \end{bmatrix} \in L_2(0,1) \oplus L_2(0,1)$, there exists a weak solution to Equations (7.10)-(7.11).

For later use, let $e = \hat{w} - w$ denote the state estimation error. Then, from Equation (7.11), the boundary conditions for the error variable e can be obtained as

$$\nu_1 e(0,t) + \nu_2 e_x(0,t) = 0 \quad \text{and} \quad \rho_1 e(1,t) + \rho_2 e_x(1,t) = q(t). \tag{7.14}$$

For these boundary conditions, we provide the following definition analogous to Definition 6.2.

Definition 7.2. *Given scalars ν_1, ν_2, ρ_1 and ρ_2, we define*

$$\{l_1, l_2, l_3\} = \begin{cases} \{-\frac{\nu_1}{\nu_2}, 0, 1\} & \textit{if } \nu_1, \nu_2 \neq 0 \\ \{0, 1, 0\} & \textit{if } \nu_1 \neq 0, \nu_2 = 0 \\ \{0, 0, 1\} & \textit{if } \nu_1 = 0, \nu_2 \neq 0 \end{cases} \cdot$$

With this definition, the boundary condition at $x = 0$ given in Equation (7.11) *can be represented as*

$$e_x(0,t) = l_1 e(0,t) + l_2 e_x(0,t), \quad e(0) = l_3 e(0,t).$$

7.1 Observer Design

In this section we wish to design observers such that its state estimates the state of the plant to be controlled with an exponentially vanishing error. Then, in the following section, we show that this observer can be coupled to the controllers designed in Theorem 6.4 to produce an exponentially stabilizing observer based boundary controller.

We begin by defining the state estimation error $e(x,t) = \hat{w}(x,t) = w(x,t)$, the dynamics of which can be obtained from Equations (7.10)-(7.11) as

$$e_t(x,t) = a(x)e_{xx}(x,t) + b(x)e_x(x,t) + c(x)e(x,t) + p(x,t), \tag{7.15}$$

with boundary conditions

$$\nu_1 e(0,t) + \nu_2 e_x(0,t) = 0 \quad \text{and} \quad \rho_1 e(1,t) + \rho_2 e_x(1,t) = q(t). \tag{7.16}$$

The main result depends primarily on the following upper bound - the proof of which can be found in Corollary B.5 in Appendix B.

$$\begin{aligned}
&\langle \mathcal{A}e(\cdot,t), \mathcal{P}e(\cdot,t)\rangle + \langle e(\cdot,t), \mathcal{P}\mathcal{A}e(\cdot,t)\rangle \\
&\le \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + e_x(0,t)\int_0^1 R_3(x)e(x,t)dx \\
&+ e(0,t)\left(R_4 e(0,t) + R_5 e_x(0,t) + \int_0^1 R_6(x)e(x,t)dx\right) \\
&+ e(1,t)\left(R_7 e(1,t) + R_8 e_x(1,t) + \int_0^1 R_9(x)e(x,t)dx\right) \\
&+ e_x(1,t)\int_0^1 R_{10}(x)e(x,t)dx,
\end{aligned}$$

where $e(\cdot,t)$ is any solution of Equations (7.15)-(7.16),

$$(\mathcal{P}y)(x) = N(x)y(x) + \int_0^x L_1(x,\xi)y(\xi)d\xi + \int_x^1 L_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1),$$

and we define the operator $\mathcal{R}$ as

$$(\mathcal{R}y)(x) = R_0(x)y(x) + \int_0^x R_1(x,\xi)y(\xi)d\xi + \int_x^1 R_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1),$$

where

$$\{R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}\} = \mathcal{J}(N, L_1, L_2)$$

and the linear operator $\mathcal{J}$ is defined as follows.

Definition 7.3. *We say*

$$\{R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}\} = \mathcal{J}(N, L_1, L_2)$$

if the following hold

$$R_0(x) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(a(x)N(x)) - b(x)N(x)\right) + 2N(x)c(x) - \frac{\alpha\epsilon\pi^2}{2}$$

$$
\begin{aligned}
&+2\left[\frac{\partial}{\partial x}\left[a(x)\left(L_1(x,\xi)-L_2(x,\xi)\right)\right]\right]_{\xi=x},\\
R_1(x,\xi)=&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[a(x)L_1(x,\xi)\right]-b(x)L_1(x,\xi)\right)+c(x)L_1(x,\xi)\\
&+\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left[a(\xi)L_1(x,\xi)\right]-b(\xi)L_1(x,\xi)\right)+c(\xi)L_1(x,\xi),\\
R_2(x,\xi)=&\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[a(x)L_2(x,\xi)\right]-b(x)L_2(x,\xi)\right)+c(x)L_2(x,\xi)\\
&+\frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left[a(\xi)L_2(x,\xi)\right]-b(\xi)L_2(x,\xi)\right)+c(\xi)L_2(x,\xi),\\
R_3(x)=&-2l_2a(0)L_2(0,x),\\
R_4=&-2l_3l_1a(0)N(0)\\
&+l_3^2\left[a_x(0)N(0)+a(0)N_x(0)-b(0)N(0)+\frac{\alpha\epsilon\pi^2}{2}\right],\\
R_5=&-2l_3n_2a(0)N(0),\\
R_6(x)=&-L_2(0,x)\left[2l_1a(0)+2l_3b(0)\right]+2l_3\left[a_x(0)L_2(0,x)+a(0)L_{2,x}(0,x)\right],\\
R_7=&-a_x(1)N(1)-a(1)N_x(1)+b(1)N(1),\\
R_8=&2a(1)N(1),\\
R_9(x)=&-2a_x(1)L_1(1,x)-2a(1)L_{1,x}(1,x)+2b(1)L_1(1,x),\\
R_{10}(x)=&2a(1)L_1(1,x),
\end{aligned}
$$

where $L_{1,x}(1,x)=\left[L_{1,x}(x,\xi)|_{x=1}\right]_{\xi=x}$, $L_{2,x}(0,x)=\left[L_{2,x}(x,\xi)|_{x=0}\right]_{\xi=x}$ *and* $\epsilon>0$ *and* l_i, $i\in\{1,\cdots,3\}$, *are scalars.*

We present the following theorem.

Theorem 7.4. *Suppose that there exist scalars* $\epsilon,\delta>0$ *and* $\{N,L_1,L_2\}\in\Xi_{d_1,d_2,\epsilon}$ *such that*

$$
\begin{aligned}
&\{-R_0-2\delta N,-R_1-2\delta L_1,-R_2-2\delta L_2\}\in\Xi_{d_1,d_2,0},\\
&R_3(x)=R_5=R_6(x)=0,\quad R_4\le 0,,
\end{aligned}
$$

for all l_j, $j \in \{1, \cdots, 3\}$ where l_j are given by Definition 7.2 and

$$\{R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}\} = \mathcal{J}(N, L_1, L_2).$$

Define the operator $\mathcal{O} := \mathcal{P}^{-1}\mathcal{V}$ where, for any $\kappa \in \mathbb{R}$,

$$(\mathcal{V}\kappa)(x) = \begin{cases} V_1(x)\kappa = -\frac{1}{2\mu_1}\left(R_9(x) + \frac{O\mu_1}{\rho_2}R_{10}(x)\right)\kappa, & \rho_1 = 0, \rho_2 \neq 0 \\ V_2(x)\kappa = -\frac{1}{2\mu_2}\left(\frac{O\mu_2}{\rho_1}R_9(x) + R_{10}(x)\right)\kappa, & \rho_1 \neq 0, \rho_2 = 0 \\ V_3(x)\kappa = -\frac{1}{2\mu_1}\left(R_9(x) + \left(\frac{O\mu_1 - \rho_1}{\rho_2}\right)R_{10}(x)\right)\kappa, & \rho_1 \neq 0, \rho_2 \neq 0 \end{cases},$$

and O is any scalar that satisfies $O < 0$ and

$$\begin{aligned} &O < -\rho_2 R_7/\mu_1 R_8 \quad \text{when} && \rho_1 = 0, \rho_2 \neq 0, \\ &\frac{1}{O} < -\mu_2 R_7/\rho_1 R_8 \quad \text{when} && \rho_1 \neq 0, \rho_2 = 0, \\ &O < \rho_1/\mu_1 - \rho_2 R_7/\mu_1 R_8 \quad \text{when} && \rho_1 \neq 0, \rho_2 \neq 0. \end{aligned}$$

Additionally,

$$(\mathcal{P}y)(x) = N(x)y(x) + \int_0^x L_1(x,\xi)y(\xi)d\xi + \int_x^1 L_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1).$$

Then for any solution $\hat{w}$ of (7.8)- (7.9) with $p(\cdot,t) = \mathcal{O}(\hat{y}(t) - y(t))$ and $q(t) = O(\hat{y}(t) - y(t))$ and any solution w of (7.1)- (7.2), there exists a scalar $M \geq 0$ such that

$$\|e(\cdot,t)\| \leq e^{-\delta t}M, \quad t \geq 0,$$

where $e = \hat{w} - w$ and $e_0 = \hat{w}_0 - w_0$ and the initial conditions satisfy

$$\begin{bmatrix} w_0 \\ \hat{w}_0 \end{bmatrix} \in \hat{\mathcal{D}},$$

for any $\mathcal{F} : H^2(0,1) \to \mathbb{R}$, and the space $\hat{\mathcal{D}}$ is defined in Equation (7.13).

Proof. Consider the Lyapunov function $V(e(\cdot,t)) = \langle e(\cdot,t), \mathcal{P}e(\cdot,t)\rangle$, where $e(x,t) = \hat{w}(x,t) - w(x,t)$ is the state estimation error whose dynamics are governed by Equations (7.15)-(7.16). Taking the derivative along the trajectories of the system, we have

$$\begin{aligned}\frac{d}{dt}V(e(\cdot,t)) &= \langle e_t(\cdot,t), \mathcal{P}e(\cdot,t)\rangle + \langle e(\cdot,t), \mathcal{P}e_t(\cdot,t)\rangle \\ &= \langle \mathcal{A}e(\cdot,t), \mathcal{P}e(\cdot,t)\rangle + \langle e(\cdot,t), \mathcal{P}\mathcal{A}e(\cdot,t)\rangle + 2\langle \mathcal{P}e(\cdot,t), p(\cdot,t)\rangle,\end{aligned}$$

where we have used the fact that $\mathcal{P}$ is self-adjoint. Using Corollary B.5,

$$\begin{aligned}&\frac{d}{dt}V(e(\cdot,t)) \\ &\le \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + e_x(0,t)\int_0^1 R_3(x)e(x,t)dx \\ &\quad + e(0,t)\left(R_4e(0,t) + R_5e_x(0,t) + \int_0^1 R_6(x)e(x,t)dx\right) \\ &\quad + e(1,t)\left(R_7e(1,t) + R_8e_x(1,t) + \int_0^1 R_9(x)e(x,t)dx\right) \\ &\quad + e_x(1,t)\int_0^1 R_{10}(x)e(x,t)dx + 2\langle \mathcal{P}e(\cdot,t), p(\cdot,t)\rangle.\end{aligned}$$

Since from the theorem statement $R_3(x) = R_5 = R_6(x) = 0$ and $R_4 \le 0$, thus

$$\begin{aligned}\frac{d}{dt}V(e(\cdot,t)) \le{}& \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle \\ &+ e(1,t)\left(R_7e(1,t) + R_8e_x(1,t) + \int_0^1 R_9(x)e(x,t)dx\right) \\ &+ e_x(1,t)\int_0^1 R_{10}(x)e(x,t)dx + 2\langle \mathcal{P}e(\cdot,t), p(\cdot,t)\rangle. \end{aligned} \tag{7.17}$$

Now,

$$p(x,t) = \left(\mathcal{O}(\hat{y}(t) - y(t))\right)(x).$$

Thus,

$$\begin{aligned}\langle \mathcal{P}e(\cdot,t), p(\cdot,t)\rangle &= \langle \mathcal{P}e(\cdot,t), \mathcal{O}\left(\hat{y}(t) - y(t)\right)\rangle \\ &= \langle e(\cdot,t), \mathcal{P}\mathcal{O}\left(\hat{y}(t) - y(t)\right)\rangle,\end{aligned}$$

where we have utilized the fact that $\mathcal{P}$ is self-adjoint. Since $\mathcal{O} = \mathcal{P}^{-1}\mathcal{V}$, we have that $\mathcal{P}\mathcal{O} = \mathcal{V}$. Thus,

$$\langle \mathcal{P}e(\cdot,t), p(\cdot,t)\rangle = \langle e(\cdot,t), \mathcal{P}\mathcal{O}\left(\hat{y}(t) - y(t)\right)\rangle = \langle e(\cdot,t), \mathcal{V}\left(\hat{y}(t) - y(t)\right)\rangle .$$

Substituting into Equation (7.17) produces

$$\begin{aligned}\frac{d}{dt}V_o(e(\cdot,t)) \leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + 2\left\langle e(\cdot,t), \mathcal{V}\left(\hat{y}(t) - y(t)\right)\right\rangle \\ + e(1,t)\left(R_7 e(1,t) + R_8 e_x(1,t) + \int_0^1 R_9(x)e(x,t)dx\right) \\ + e_x(1,t)\int_0^1 R_{10}(x)e(x,t)dx.\end{aligned} \tag{7.18}$$

From the condition in Equation (7.5) we have that

$$|\rho_1| + |\rho_2| > 0.$$

Thus, there are three possible cases:

$$\begin{aligned} &\text{CASE 1:} \quad \rho_1 = 0, \quad \rho_2 \neq 0, \\ &\text{CASE 2:} \quad \rho_1 \neq 0, \quad \rho_2 = 0, \\ &\text{CASE 3:} \quad \rho_1 \neq 0, \quad \rho_2 \neq 0. \end{aligned}$$

For the case when $\rho_1 = 0$ and $\rho_2 \neq 0$, we have that

$$\rho_2 e_x(1,t) = q(t)$$

or

$$e_x(1,t) = \frac{1}{\rho_2}O(\hat{y}(t) - y(t)).$$

From Equation (7.6), when $\rho_1 = 0$, we have that $\mu_1 \neq 0$ and $\mu_2 = 0$. Thus

$$\hat{y}(t) - y(t) = \mu_1 e(1,t).$$

Thus

$$e_x(1,t) = \frac{O\mu_1}{\rho_2}e(1,t). \tag{7.19}$$

Moreover,

$$\left(\mathcal{V}(\hat{y}(t) - y(t))\right)(x) = \mu_1 \left(\mathcal{V}e(1,t)\right)(x). \tag{7.20}$$

Substituting Equations (7.19)-(7.20) into Equation (7.18) and collecting terms produces

$$\begin{aligned}
&\frac{d}{dt}V(e(\cdot,t)) \\
&\leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + 2\mu_1 \langle e(\cdot,t), \mathcal{V}e(1,t)\rangle \\
&\quad + e^2(1,t)\left(R_7 + \frac{O\mu_1}{\rho_2}R_8\right) + e(1,t)\int_0^1 \left(R_9(x) + \frac{O\mu_1}{\rho_2}R_{10}(x)\right) e(x,t)dx.
\end{aligned} \tag{7.21}$$

From the theorem statement, when $\rho_1 = 0$ and $\rho_2 \neq 0$

$$O < 0 \quad \text{and} \quad O < -\frac{\rho_2 R_7}{\mu_1 R_8},$$

which is well defined as $R_8 = 2a(1)N(1) > 0$. Thus there exists a scalar $\omega_1 > 0$ such that

$$R_7 + \frac{O\mu_1}{\rho_2}R_8 = -\,\omega_1. \tag{7.22}$$

Additionally

$$\left(\mathcal{V}\kappa\right)(x) = V_1(x)\kappa = -\frac{1}{2\mu_1}\left(R_9(x) + \frac{O\mu_1}{\rho_2}R_{10}(x)\right)\kappa,$$

for any $\kappa \in \mathbb{R}$. Thus

$$2\mu_1 \langle e(\cdot,t), \mathcal{V}e(1,t)\rangle = -e(1,t)\int_0^1 \left(R_9(x) + \frac{O\mu_1}{\rho_2}R_{10}(x)\right) e(x,t)dx. \tag{7.23}$$

Substituting Equations (7.22)-(7.23) into Equation (7.21) produces

$$\frac{d}{dt}V_o(e(\cdot,t)) \leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle - \omega_1 e(1,t)^2, \tag{7.24}$$

when $\rho_1 = 0$ and $\rho_2 \neq 0$ for some $\omega_1 > 0$.

For the case when $\rho_1 \neq 0$ and $\rho_2 = 0$, we have that

$$\rho_1 e(1,t) = q(t),$$

or

$$e(1,t) = \frac{1}{\rho_1} O(\hat{y}(t) - y(t)).$$

From Equation (7.6), when $\rho_1 \neq 0$ and $\rho_2 = 0$, $\mu_1 = 0$ and $\mu_2 \neq 0$. Thus,

$$\hat{y}(t) - y(t) = \mu_2 e_x(1,t).$$

Thus,

$$e(1,t) = \frac{O\mu_2}{\rho_1} e_x(1,t), \tag{7.25}$$

and

$$e_x(1,t) = \frac{\rho_1}{O\mu_2} e(1,t), \tag{7.26}$$

which is well defined since for this case $O \neq 0$. Moreover

$$\left(\mathcal{V}(\hat{y}(t) - y(t))\right)(x) = \mu_2 \left(\mathcal{V} e_x(1,t)\right)(x). \tag{7.27}$$

Substituting Equations (7.25)-(7.27) into Equation (7.18) produces

$$\begin{aligned}
&\frac{d}{dt} V(e(\cdot,t)) \\
&\leq \langle e(\cdot,t), \mathcal{R} e(\cdot,t) \rangle + 2\mu_2 \langle e(\cdot,t), \mathcal{V} e_x(1,t) \rangle \\
&\quad + e(1,t)^2 \left(R_7 + \frac{\rho_1}{O\mu_2} R_8 \right) + e_x(1,t) \int_0^1 \left(\frac{O\mu_2}{\rho_1} R_9(x) + R_{10}(x) \right) e(x,t) dx.
\end{aligned} \tag{7.28}$$

From the theorem statement, when $\rho_1 \neq 0$ and $\rho_2 = 0$

$$O < 0 \quad \text{and} \quad \frac{1}{O} < -\frac{\mu_2}{\rho_1} \frac{R_7}{R_8}$$

Thus, there exists a scalar $\omega_2 > 0$ such that

$$R_7 + \frac{\rho_1}{O\mu_2} R_8 = -\omega_2, \tag{7.29}$$

since $R_8 = 2a(1)N(1) > 0$. Substituting (7.29) in (7.28) produces,

$$\begin{aligned}
&\frac{d}{dt} V(e(\cdot,t)) \\
&\leq \langle e(\cdot,t), \mathcal{R} e(\cdot,t) \rangle + 2\mu_2 \langle e(\cdot,t), \mathcal{V} e_x(1,t) \rangle
\end{aligned}$$

$$- \omega_2 e(1,t)^2 + e_x(1,t) \int_0^1 \left(\frac{O\mu_2}{\rho_1} R_9(x) + R_{10}(x) \right) e(x,t) dx. \tag{7.30}$$

Moreover, from the theorem statement,

$$\left(\mathcal{V}\kappa\right)(x) = V_2(x)\kappa = -\frac{1}{2\mu_2} \left(\frac{O\mu_2}{\rho_1} R_9(x) + R_{10}(x) \right) \kappa,$$

for any $\kappa \in \mathbb{R}$. Thus,

$$2\mu_2 \left\langle e(\cdot,t), \mathcal{V} e_x(1,t) \right\rangle = -e_x(1,t) \int_0^1 \left(\frac{O\mu_2}{\rho_1} R_9(x) + R_{10}(x) \right) e(x,t) dx. \tag{7.31}$$

Substituting Equation (7.31) into Equation (7.30) produces

$$\frac{d}{dt} V(e(\cdot,t)) \leq \left\langle e(\cdot,t), \mathcal{R} e(\cdot,t) \right\rangle - \omega_2 e(1,t)^2, \tag{7.32}$$

when $\rho_1 \neq 0$ and $\rho_2 = 0$ for some $\omega_2 > 0$.

For the case when $\rho_1 \neq 0$ and $\rho_2 \neq 0$, we have that

$$\rho_1 e(1,t) + \rho_2 e_x(1,t) = q(t),$$

or

$$e_x(1,t) = \frac{1}{\rho_2} O(\hat{y}(t) - y(t)) - \frac{\rho_1}{\rho_2} e(1,t).$$

From Equation (7.6), when $\rho_1 \neq 0$ and $\rho_2 \neq 0$, $\mu_1 \neq 0$ and $\mu_2 = 0$. Thus,

$$\hat{y}(t) - y(t) = \mu_1 e(1,t).$$

Thus,

$$e_x(1,t) = \left(\frac{O\mu_1 - \rho_1}{\rho_2} \right) e(1,t). \tag{7.33}$$

Moreover

$$\left(\mathcal{V}(\hat{y}(t) - y(t))\right)(x) = \mu_1 \left(\mathcal{V} e(1,t)\right)(x). \tag{7.34}$$

Substituting Equations (7.33)-(7.34) into Equation (7.18) produces

$$\frac{d}{dt} V(e(\cdot,t))$$

$$\leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + 2\mu_1 \langle e(\cdot,t), \mathcal{V}e(1,t)\rangle + e^2(1,t)\left(R_7 + \left(\frac{O\mu_1 - \rho_1}{\rho_2}\right) R_8\right)$$
$$+ e(1,t)\int_0^1 \left(R_9(x) + \left(\frac{O\mu_1 - \rho_1}{\rho_2}\right) R_{10}(x)\right) e(x,t)dx. \tag{7.35}$$

From the theorem statement, when $\rho_1 \neq 0$ and $\rho_2 \neq 0$,

$$O < 0 \quad \text{and} \quad O < \frac{\rho_1}{\mu_1} - \frac{\rho_2 R_7}{\mu_1 R_8},$$

which is well defined as $R_8 = 2a(1)N(1) > 0$. Thus, there exists a scalar $\omega_3 > 0$ such that

$$R_7 + \left(\frac{O\mu_1 - \rho_1}{\rho_2}\right) R_8 = -\omega_3. \tag{7.36}$$

Additionally,

$$\left(\mathcal{V}\kappa\right)(x) = V_3(x)\kappa = -\frac{1}{2\mu_1}\left(R_9(x) + \left(\frac{O\mu_1 - \rho_1}{\rho_2}\right) R_{10}(x)\right)\kappa,$$

for any $\kappa \in \mathbb{R}$. Thus,

$$2\mu_1 \langle e(\cdot,t), \mathcal{V}e(1,t)\rangle = -e(1,t)\int_0^1 \left(R_9(x) + \left(\frac{O\mu_1 - \rho_1}{\rho_2}\right) R_{10}(x)\right) e(x,t)dx. \tag{7.37}$$

Substituting Equations (7.36)-(7.37) into Equation (7.35) produces

$$\frac{d}{dt}V(e(\cdot,t)) \leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle - \omega_3 e(1,t)^2, \tag{7.38}$$

when $\rho_1 \neq 0$ and $\rho_2 \neq 0$ for some $\omega_3 > 0$.

From Equations (7.24), (7.32) and (7.38) we conclude that for any $\rho_1, \rho_2 \in \mathbb{R}$ which satisfy

$$|\rho_1| + |\rho_2| > 0,$$

there exists scalars $\omega_1, \omega_2, \omega_3 > 0$ such that

$$\frac{d}{dt}V(e(\cdot,t)) \leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle - \omega e(1,t)^2, \tag{7.39}$$

where $\omega = \min\{\omega_1, \omega_2, \omega_3\}$.

Since $\omega > 0$, we conclude that

$$\frac{d}{dt}V(e(\cdot,t)) \leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle . \tag{7.40}$$

From the theorem statement we have that

$$\{-R_0 - 2\delta N, -R_1 - 2\delta L_1, -R_2 - 2\delta L_2\} \in \Xi_{d_1,d_2,0},$$

and hence, from Equation (7.40), we conclude that

$$\frac{d}{dt}V(e(\cdot,t)) \leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle \leq -2\delta \langle e(\cdot,t), \mathcal{P}e(\cdot,t)\rangle .$$

Therefore,

$$\frac{d}{dt}V(e(\cdot,t)) \leq -2\delta V(e(\cdot,t)), \quad t \geq 0.$$

Integrating in time yields

$$V(e(\cdot,t)) = \langle e(\cdot,t), (\mathcal{P}e)(\cdot,t)\rangle \leq e^{-2\delta t}\langle e_0, \mathcal{P}e_0\rangle,$$

and since, $\{N, L_1, L_2\} \in \Xi_{d_1,d_2,\epsilon}$, we have

$$\epsilon\|e(\cdot,t)\|^2 \leq \langle e(\cdot,t), (\mathcal{P}e)(\cdot,t)\rangle \leq e^{-2\delta t}\langle e_0, \mathcal{P}e_0\rangle, \quad t \geq 0$$

which implies

$$\|e(\cdot,t)\| \leq e^{-\delta t}\sqrt{\frac{\langle e_0, \mathcal{P}e_0\rangle}{\epsilon}}, \quad t \geq 0.$$

Setting

$$M = \sqrt{\frac{\langle e_0, \mathcal{P}e_0\rangle}{\epsilon}}$$

completes the proof. □

7.2 Exponentially Stabilizing Observer Based Boundary Control

We now prove that the observer designed in Theorem 7.4 can be coupled to the controlled designed in Theorem 6.4 to produce an exponentially stabilizing observer based feedback controller. This is known as the *separation principle* [36].

Theorem 7.5. *Suppose that there exist scalars $\epsilon, \delta_c, \delta_o > 0$, $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ and $\{N, L_1, L_2\} \in \Xi_{d_1,d_2,\epsilon}$, such that*

$$\begin{aligned}
&\{-T_0 - 2\delta_c M, -T_1 - 2\delta_c K_1, -T_2 - 2\delta_c K_2\} \in \Xi_{d_1,d_2,0},\\
&\{-R_0 - 2\delta_o N, -R_1 - 2\delta_o L_1, -R_2 - 2\delta_o L_2\} \in \Xi_{d_1,d_2,0},\\
&T_3 \le 0, \quad T_4(x) = T_5(x) = T_6(x) = 0,\\
&R_4 \le 0, \quad R_3(x) = R_5 = R_6(x) = 0,
\end{aligned}$$

for all l_j, $j \in \{1, \cdots, 3\}$ where l_j are given by Definition 7.2 and for all m_j, $j \in \{1, \cdots, 3\}$ where m_j are given by Definition 6.2. Here,

$$\begin{aligned}
\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} =& \mathcal{N}(M, K_1, K_2),\\
\{R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}\} =& \mathcal{J}(N, L_1, L_2).
\end{aligned}$$

Define the operator $\mathcal{F} := \mathcal{Z}\mathcal{P}_c^{-1}$ where, for any $y \in H^2(0,1)$,

$$\mathcal{Z}y = \begin{cases} Z_1 y(1) + \int_0^1 Z_2(x) y(x) dx & \rho_1 = 0, \rho_2 \neq 0 \\ Z_3 y_x(1) + \int_0^1 Z_4(x) y(x) dx & \rho_1 \neq 0, \rho_2 = 0 \\ Z_5 y(1) + \int_0^1 Z_6(x) y(x) dx & \rho_1 \neq 0, \rho_2 \neq 0 \end{cases} .$$

Here, Z_1, Z_3 and Z_5 are any scalars that satisfy

$$\begin{aligned}
&Z_1 < 0 \quad \text{and} \quad Z_1 < -\frac{\rho_2}{2a(1)} \left(T_7 - 2a(1) M_x(1)\right),\\
&Z_3 < 0 \quad \text{and} \quad \frac{1}{Z_3} < -\frac{1}{\rho_1 M(1)} \frac{T_7}{T_8},\\
&Z_5 < 0 \quad \text{and} \quad Z_5 < -\frac{\rho_2}{2a(1)} \left(T_7 - \frac{\rho_1}{\rho_2} T_8 - 2a(1) M_x(1)\right),
\end{aligned}$$

and polynomials $Z_2(x)$, $Z_4(x)$ and $Z_6(x)$ are defined as

$$\begin{aligned}
&Z_2(x) = \rho_2 K_{1,x}(1, x),\\
&Z_4(x) = \rho_1 K_1(1, x),
\end{aligned}$$

$$Z_6(x) = \rho_2 \left(\frac{\rho_1}{\rho_2} K_1(1,x) + K_{1,x}(1,x) \right).$$

Additionally, define the operator $\mathcal{O} := \mathcal{P}_o^{-1}\mathcal{V}$ *where, for any* $\kappa \in \mathbb{R}$,

$$(\mathcal{V}\kappa)(x) = \begin{cases} V_1(x)\kappa = -\frac{1}{2\mu_1}\left(R_9(x) + \frac{O\mu_1}{\rho_2}R_{10}(x)\right)\kappa, & \rho_1 = 0, \rho_2 \neq 0 \\ V_2(x)\kappa = -\frac{1}{2\mu_2}\left(\frac{O\mu_2}{\rho_1}R_9(x) + R_{10}(x)\right)\kappa, & \rho_1 \neq 0, \rho_2 = 0 \\ V_3(x)\kappa = -\frac{1}{2\mu_1}\left(R_9(x) + \left(\frac{O\mu_1 - \rho_1}{\rho_2}\right)R_{10}(x)\right)\kappa, & \rho_1 \neq 0, \rho_2 \neq 0 \end{cases},$$

and O *is any scalar that satisfies* $O < 0$ *and*

$$\begin{aligned} &O < -\rho_2 R_7/\mu_1 R_8 \quad \text{when} \quad && \rho_1 = 0, \rho_2 \neq 0, \\ &\frac{1}{O} < -\mu_2 R_7/\rho_1 R_8 \quad \text{when} \quad && \rho_1 \neq 0, \rho_2 = 0, \\ &O < \rho_1/\mu_1 - \rho_2 R_7/\mu_1 R_8 \quad \text{when} \quad && \rho_1 \neq 0, \rho_2 \neq 0. \end{aligned}$$

Moreover, for any $y \in L_2(0,1)$,

$$\begin{aligned} (\mathcal{P}_c y)(x) =& M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \\ (\mathcal{P}_o y)(x) =& N(x)y(x) + \int_0^x L_1(x,\xi)y(\xi)d\xi + \int_x^1 L_2(x,\xi)y(\xi)d\xi. \end{aligned}$$

Then, for any solution w *of* (7.1)- (7.2) *with* $u(t) = \mathcal{F}\hat{w}(\cdot,t)$, *where* $\hat{w}$ *is a solution of* (7.8)- (7.9) *with* $p(\cdot,t) = \mathcal{O}(\hat{y}(t) - y(t))$ *and* $q(t) = O(\hat{y}(t) - y(t))$, *there exists a scalar* $M \geq 0$ *such that*

$$\|w(\cdot,t)\| \leq e^{-\delta t}M, \quad t \geq 0,$$

for any $0 < \delta < \min\{\delta_c, \delta_o\}$.

Proof. Consider the Lyapunov function $V_o(e(\cdot,t)) = \langle e(\cdot,t), \mathcal{P}_o e(\cdot,t)\rangle$, where $e(x,t) = \hat{w}(x,t) - w(x,t)$ is the state estimation error whose dynamics are governed by Equations (7.15)-(7.16). Taking the derivative along the trajectories of the system, we have

$$\frac{d}{dt}V_o(e(\cdot,t)) = \langle e_t(\cdot,t), \mathcal{P}_o e(\cdot,t)\rangle + \langle e(\cdot,t), \mathcal{P}_o e_t(\cdot,t)\rangle$$
$$= \langle \mathcal{A}e(\cdot,t), \mathcal{P}_o e(\cdot,t)\rangle + \langle e(\cdot,t), \mathcal{P}_o\mathcal{A}e(\cdot,t)\rangle + 2\left\langle \mathcal{P}_o e(\cdot,t), p(\cdot,t)\right\rangle,$$

where we have used the fact that $\mathcal{P}_o$ is self-adjoint. Using Corollary B.5,

$$\begin{aligned}
&\frac{d}{dt}V_o(e(\cdot,t))\\
&\leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + e_x(0,t)\int_0^1 R_3(x)e(x,t)dx\\
&\quad + e(0,t)\left(R_4 e(0,t) + R_5 e_x(0,t) + \int_0^1 R_6(x)e(x,t)dx\right)\\
&\quad + e(1,t)\left(R_7 e(1,t) + R_8 e_x(1,t) + \int_0^1 R_9(x)e(x,t)dx\right)\\
&\quad + e_x(1,t)\int_0^1 R_{10}(x)e(x,t)dx + 2\left\langle \mathcal{P}_o e(\cdot,t), p(\cdot,t)\right\rangle.
\end{aligned}$$

From the theorem statement we have that $R_3(x) = R_5 = R_6(x) = 0$ and $R_4 \leq 0$, therefore

$$\begin{aligned}
&\frac{d}{dt}V_o(e(\cdot,t))\\
&\leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + 2\left\langle \mathcal{P}_o e(\cdot,t), p(\cdot,t)\right\rangle\\
&\quad + e(1,t)\left(R_7 e(1,t) + R_8 e_x(1,t) + \int_0^1 R_9(x)e(x,t)dx\right)\\
&\quad + e_x(1,t)\int_0^1 R_{10}(x)e(x,t)dx.
\end{aligned}$$

With the operator $\mathcal{O}$ and scalar O as defined in the theorem statement, using the analysis presented in Theorem 7.4 and from Equation (7.39), we conclude that there exists a scalar $\omega > 0$ such that

$$\frac{d}{dt}V_o(e(\cdot,t)) \leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle - \omega e(1,t)^2. \tag{7.41}$$

Now recall the dynamics of the observer given by

$$\hat{w}_t(x,t) = a(x)\hat{w}_{xx}(x,t) + b(x)\hat{w}_x(x,t) + c(x)\hat{w}(x,t) + p(x,t), \tag{7.42}$$

$$\nu_1 \hat{w}(0,t) + \nu_2 \hat{w}_x(0,t) = 0, \quad \rho_1 \hat{w}(1,t) + \rho_2 \hat{w}_x(1,t) = q(t) + u(t). \tag{7.43}$$

For the observer, consider the following Lyapunov function

$$V_c(\hat{w}(\cdot,t)) = \left\langle \hat{w}(\cdot,t), \mathcal{P}_c^{-1} \hat{w}(\cdot,t) \right\rangle.$$

Taking the time derivative along trajectories of the system, we have

$$\begin{aligned}\frac{d}{dt} V_c(\hat{w}(\cdot,t)) =& \left\langle \mathcal{A}\hat{w}(\cdot,t), \mathcal{P}_c^{-1}\hat{w}(\cdot,t) \right\rangle + \left\langle \mathcal{P}_c^{-1}\hat{w}(\cdot,t), \mathcal{A}\hat{w}(\cdot,t) \right\rangle \\ &+ 2\left\langle \mathcal{P}_c^{-1}\hat{w}(\cdot,t), p(\cdot,t) \right\rangle,\end{aligned}$$

where we have used the fact that $\mathcal{P}_c = \mathcal{P}_c^{\star}$ implies $\mathcal{P}_c^{-1} = (\mathcal{P}_c^{\star})^{-1}$.

Now let $\hat{z} = \mathcal{P}_c^{-1}\hat{w}$. Then

$$\begin{aligned}\frac{d}{dt} V_c(\hat{w}(\cdot,t)) =& \left\langle \mathcal{A}\mathcal{P}_c\mathcal{P}_c^{-1}\hat{w}(\cdot,t), \mathcal{P}_c^{-1}\hat{w}(\cdot,t) \right\rangle + \left\langle \mathcal{P}_c^{-1}\hat{w}(\cdot,t), \mathcal{A}\mathcal{P}_c\mathcal{P}_c^{-1}\hat{w}(\cdot,t) \right\rangle \\ &+ 2\left\langle \mathcal{P}_c^{-1}\hat{w}(\cdot,t), p(\cdot,t) \right\rangle \\ =& \left\langle \mathcal{A}\mathcal{P}_c\hat{z}(\cdot,t), \hat{z}(\cdot,t) \right\rangle + \left\langle \hat{z}(\cdot,t), \mathcal{A}\mathcal{P}_c\hat{z}(\cdot,t) \right\rangle + 2\left\langle \hat{z}(\cdot,t), p(\cdot,t) \right\rangle.\end{aligned}$$

From Corollary B.7,

$$\begin{aligned}&\frac{d}{dt} V_c(\hat{w}(\cdot,t)) \\ &\leq \left\langle \hat{z}(\cdot,t), \mathcal{T}\hat{z}(\cdot,t) \right\rangle + 2\left\langle \hat{z}(\cdot,t), p(\cdot,t) \right\rangle \\ &\quad + \hat{z}(0,t)\left(T_3\hat{z}(0,t) + \int_0^1 T_4(x)\hat{z}(x,t)dx \right) + \hat{z}_x(0,t)\int_0^1 T_5(x)\hat{z}(x,t)dx \\ &\quad + \int_0^1 \frac{1}{M(0)} T_6(x)\hat{z}(x,t)dx \left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2 \right)\hat{z}(0,t) \\ &\quad + \int_0^1 \frac{1}{M(0)} T_6(x)\hat{z}(x,t)dx \int_0^1 \alpha\epsilon\pi^2 \hat{z}(x,t)dx \\ &\quad + \hat{z}(1,t)\left(T_7\hat{z}(1,t) + T_8\hat{z}_x(1,t) \right).\end{aligned}$$

From the theorem statement we have that $T_4(x) = T_5(x) = T_6(x) = 0$ and $T_3 \leq 0$, therefore

$$\frac{d}{dt} V_c(\hat{w}(\cdot,t)) \leq \left\langle \hat{z}(\cdot,t), \mathcal{T}\hat{z}(\cdot,t) \right\rangle + 2\left\langle \hat{z}(\cdot,t), p(\cdot,t) \right\rangle + \hat{z}(1,t)\left(T_7\hat{z}(1,t) + T_8\hat{z}_x(1,t) \right). \tag{7.44}$$

Now, from the theorem statement $u(t) = \mathcal{F}\hat{w}(\cdot,t)$ and $\mathcal{F} = \mathcal{Z}\mathcal{P}_c^{-1}$, which implies $\mathcal{F}\mathcal{P}_c = \mathcal{Z}$. Therefore

$$u(t) = \mathcal{F}\hat{w}(\cdot,t) = \mathcal{F}\mathcal{P}_c\mathcal{P}_c^{-1}\hat{w}(\cdot,t) = \mathcal{Z}\hat{z}(\cdot,t).$$

Thus, using (7.43), the boundary condition at $x = 1$ can be written as

$$\rho_1\hat{w}(1,t) + \rho_2\hat{w}_x(1,t) = u(t) + q(t) = \mathcal{Z}\hat{z}(\cdot,t) + q(t).$$

Using the definition of the operator $\mathcal{Z}$ from the theorem statement and applying the analysis presented in Theorem 6.4 and Equation (6.14), there exists a scalar $\zeta > 0$ such that Equation (7.44) reduces to

$$\begin{aligned} &\frac{d}{dt}V_c(\hat{w}(\cdot,t)) \\ &\le \langle \hat{z}(\cdot,t), \mathcal{T}\hat{z}(\cdot,t)\rangle - \zeta\hat{z}(1,t)^2 + 2\langle \hat{z}(\cdot,t), p(\cdot,t)\rangle + 2\hat{z}(1,t)hq(t), \end{aligned} \tag{7.45}$$

where

$$h = \begin{cases} 2a(1)/\rho_2, & \rho_1 = 0, \rho_2 \neq 0, \\ -T_8/2Z_3, & \rho_1 \neq 0, \rho_2 = 0, \\ 2a(1)/\rho_2, & \rho_1 \neq 0, \rho_2 \neq 0. \end{cases} \tag{7.46}$$

By definition $p(x,t) = (\mathcal{O}(\hat{y}(t) - y(t)))(x)$ and $\mathcal{O} = \mathcal{P}_o^{-1}\mathcal{V}$. Therefore,

$$p(x,t) = \left(\mathcal{P}_o^{-1}\mathcal{V}\left(\hat{y}(t) - y(t)\right)\right)(x).$$

Thus, using the analysis presented in Theorem 7.4 it can be established that

$$\langle \hat{z}(\cdot,t), p(\cdot,t)\rangle = e(1,t)\int_0^1 W(x)\hat{z}(x,t)dx, \tag{7.47}$$

where

$$W(x) = \begin{cases} \mu_1\left(\mathcal{P}_o^{-1}V_1\right)(x), & \rho_1 = 0, \rho_2 \neq 0, \\ (\rho_1/O)\left(\mathcal{P}_o^{-1}V_2\right)(x), & \rho_1 \neq 0, \rho_2 = 0, \\ \mu_1\left(\mathcal{P}_o^{-1}V_3\right)(x), & \rho_1 \neq 0, \rho_2 \neq 0 \end{cases}, \tag{7.48}$$

where polynomials $V_1(x)$, $V_2(x)$ and $V_3(x)$ are defined in the theorem statement.

Similarly, by definition $q(t) = O\left(\hat{y}(t) - y(t)\right)$. Thus, using the analysis presented in Theorem 7.4 it can be established that

$$\hat{z}(1,t)hq(t) = \hat{z}(1,t)ge(1,t), \tag{7.49}$$

where

$$g = \begin{cases} hO\mu_1, & \rho_1 = 0, \rho_2 \neq 0, \\ h\rho_1, & \rho_1 \neq 0, \rho_2 = 0, \\ hO\mu_1, & \rho_1 \neq 0, \rho_2 \neq 0, \end{cases}, \tag{7.50}$$

and h is defined in (7.46).

Substituting Equations (7.47) and (7.49) into (7.45) produces

$$\begin{aligned} &\frac{d}{dt}V_c(\hat{w}(\cdot,t)) \\ &\leq \langle \hat{z}(\cdot,t), \mathcal{T}\hat{z}(\cdot,t)\rangle - \zeta\hat{z}(1,t)^2 + 2e(1,t)\int_0^1 W(x)\hat{z}(x,t)dx + 2\hat{z}(1,t)ge(1,t), \end{aligned} \tag{7.51}$$

From Equations (7.41) and (7.51) we conclude that for any scalar $A > 0$,

$$\begin{aligned} &A\frac{d}{dt}V_o(e(\cdot,t)) + \frac{d}{dt}V_c(\hat{w}(\cdot,t)) \\ &\leq A\left\langle e(\cdot,t), \mathcal{R}e(\cdot,t)\right\rangle - A\omega e(1,t)^2 + \langle \hat{z}(\cdot,t), \mathcal{T}\hat{z}(\cdot,t)\rangle - \zeta\hat{z}(1,t)^2 \\ &\quad + 2e(1,t)\int_0^1 W(x)\hat{z}(x,t)dx + 2\hat{z}(1,t)ge(1,t), \end{aligned} \tag{7.52}$$

where $\zeta, \omega > 0$.

Let us define the operator $\mathcal{W} : L_2(0,1) \to L_2(0,1)$ as $\left(\mathcal{W}y\right)(x) = W(x)y(x)$, for any $y \in L_2(0,1)$. Thus, we get

$$e(1,t)\int_0^1 W(x)\hat{z}(x,t)dx = \left\langle e(1,t), \mathcal{W}\hat{z}(\cdot,t)\right\rangle.$$

Substituting into Equation (7.52) and rearranging

$$A\frac{d}{dt}V_o(e(\cdot,t)) + \frac{d}{dt}V_c(\hat{w}(\cdot,t))$$
$$\leq A\left\langle e(\cdot,t), \mathcal{R}e(\cdot,t)\right\rangle + \left\langle \begin{bmatrix} \hat{z}(\cdot,t) \\ \hat{z}(1,t) \\ e(1,t) \end{bmatrix}, \begin{bmatrix} \mathcal{T} & 0 & \mathcal{W} \\ 0 & -\zeta\mathcal{I} & g\mathcal{I} \\ \mathcal{W} & g\mathcal{I} & -A\omega\mathcal{I} \end{bmatrix} \begin{bmatrix} \hat{z}(\cdot,t) \\ \hat{z}(1,t) \\ e(1,t) \end{bmatrix} \right\rangle, \tag{7.53}$$

where $\mathcal{I}$ is the identity operator and the inner product is defined on $L_2(0,1)\oplus L_2(0,1)\oplus L_2(0,1)$.

Now, for any $0 < \theta < \delta_c$, consider the following operator on $L_2(0,1)\oplus L_2(0,1)\oplus L_2(0,1)$

$$\begin{bmatrix} \mathcal{T} + 2(\delta_c - \theta)\mathcal{P}_c & 0 & \mathcal{W} \\ 0 & -\zeta\mathcal{I} & g\mathcal{I} \\ \mathcal{W} & g\mathcal{I} & -A\omega\mathcal{I} \end{bmatrix}. \tag{7.54}$$

We can choose the scalar $A > 0$ large enough so that the operator

$$\begin{bmatrix} -\zeta\mathcal{I} & g\mathcal{I} \\ g\mathcal{I} & -A\omega\mathcal{I} \end{bmatrix} < 0$$

on $L_2(0,1) \oplus L_2(0,1)$. Therefore, we may apply Schur complements to conclude that the operator in Equation (7.54) is negative definite if and only if

$$\mathcal{T} + 2(\delta_c - \theta)\mathcal{P}_c + \frac{\zeta\hat{W}}{A\omega - g^2}\mathcal{I} < 0$$

on $L_2(0,1)$, where $\hat{W} = \sup_{x\in[0,1]} W(x)^2$. From the theorem statement

$$\{-T_0 - 2\delta_c M, -T_1 - 2\delta_c K_1, -T_2 - 2\delta_c K_2\} \in \Xi_{d_1,d_2,0}.$$

Therefore, $\mathcal{T} + 2\delta_c\mathcal{P}_c \leq 0$ and

$$\mathcal{T} + 2(\delta_c - \theta)\mathcal{P}_c + \frac{\zeta\hat{W}}{A\omega - g^2}\mathcal{I} \leq -2\theta\mathcal{P}_c + \frac{\zeta\hat{W}}{A\omega - g^2}\mathcal{I}.$$

Moreover, from the theorem statement we have that $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$. Thus, $\mathcal{P}_c \geq \epsilon\mathcal{I}$. Hence

$$\begin{aligned}\mathcal{T} + 2(\delta_c - \theta)\mathcal{P}_c + \frac{\zeta\hat{W}}{A\omega - g^2}\mathcal{I} \leq & -2\theta\mathcal{P}_c + \frac{\zeta\hat{W}}{A\omega - g^2}\mathcal{I} \\ \leq & \left(-2\theta\epsilon + \frac{\zeta\hat{W}}{A\omega - g^2}\right)\mathcal{I},\end{aligned}$$

which, for a large enough $A > 0$ is negative definite on $L_2(0,1)$. Therefore, the operator defined in Equation (7.54) is negative definite, and thus

$$\begin{bmatrix} \mathcal{T} & 0 & \mathcal{W} \\ 0 & -\zeta\mathcal{I} & g\mathcal{I} \\ \mathcal{W} & g\mathcal{I} & -A\omega\mathcal{I} \end{bmatrix} \leq \begin{bmatrix} -2(\delta_c - \theta)\mathcal{P}_c & 0 & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}.$$

Substituting into Equation (7.53)

$$\begin{aligned}& A\frac{d}{dt}V_o(e(\cdot,t)) + \frac{d}{dt}V_c(\hat{w}(\cdot,t)) \\ & \leq A\left\langle e(\cdot,t), \mathcal{R}e(\cdot,t)\right\rangle - 2(\delta_c - \theta)\left\langle \mathcal{P}_c\hat{z}(\cdot,t), \hat{z}(\cdot,t)\right\rangle.\end{aligned}$$

Since

$$\{-R_0 - 2\delta_o N, -R_1 - 2\delta_o L_1, -R_2 - 2\delta_o L_2\} \in \Xi_{d_1,d_2,0},$$

we have that $\mathcal{R} \leq -2\delta_o\mathcal{P}_o$. Thus

$$\begin{aligned}& A\frac{d}{dt}V_o(e(\cdot,t)) + \frac{d}{dt}V_c(\hat{w}(\cdot,t)) \\ & \leq -2A\delta_o\left\langle \mathcal{P}_o e(\cdot,t), e(\cdot,t)\right\rangle - 2(\delta_c - \theta)\left\langle \mathcal{P}_c\hat{z}(\cdot,t), \hat{z}(\cdot,t)\right\rangle \\ & \leq -2\delta\left(AV_o(e(\cdot,t)) + V_c(\hat{w}(\cdot,t))\right),\end{aligned} \tag{7.55}$$

where $\delta = \min\{\delta_o, \delta_c - \theta\}$. Note that since $0 < \theta < \delta_c$, $\delta > 0$. Moreover, since the presented arguments are for any arbitrary $0 < \theta < \delta_c$, we conclude that (7.55) holds for any $0 < \delta < \min\{\delta_c, \delta_o\}$.

Integrating Equation (7.55) in time yields

$$AV_o(e(\cdot,t)) + V_c(\hat{w}(\cdot,t)) \leq e^{-2\delta t}\left(AV_o(e_0) + V_c(\hat{w}_0)\right),$$

where $e_0 = e(x,0)$ and $\hat{w}_0 = \hat{w}(x,0)$.

Using the analysis presented in Theorems 5.8 and 6.4, we have that

$$\|e(\cdot,t)\|^2 \leq \frac{1}{\epsilon}V_o(e(\cdot,t)), \quad \|\hat{w}(\cdot,t)\|^2 \leq \frac{\|\mathcal{P}_c\|_{\mathcal{L}}^2}{\epsilon}V_c(\hat{w}(\cdot,t)).$$

Thus,

$$A\epsilon\|e(\cdot,t)\|^2 + \epsilon\|\mathcal{P}_c\|_{\mathcal{L}}^{-2}\|\hat{w}(\cdot,t)\| \leq e^{-2\delta t}\left(AV_o(e_0) + V_c(\hat{w}_0)\right),$$

which in turn implies

$$\begin{aligned}\|e(\cdot,t)\| \leq& \frac{1}{\sqrt{A\epsilon}}e^{-\delta t}\sqrt{AV_o(e_0) + V_c(\hat{w}_0)},\\ \|\hat{w}(\cdot,t)\| \leq& \frac{\|\mathcal{P}_c\|_{\mathcal{L}}}{\sqrt{\epsilon}}e^{-\delta t}\sqrt{AV_o(e_0) + V_c(\hat{w}_0)}.\end{aligned} \tag{7.56}$$

Since $e = \hat{w} - w$,

$$\|w(\cdot,t)\| = \|\hat{w}(\cdot,t) - e(\cdot,t)\| \leq \|\hat{w}(\cdot,t)\| + \|e(\cdot,t)\|.$$

Substituting Equation (7.56) produces,

$$\|w(\cdot,t)\| \leq e^{-\delta t}\left(\frac{1}{\sqrt{A\epsilon}} + \frac{\|\mathcal{P}_c\|_{\mathcal{L}}}{\sqrt{\epsilon}}\right)\sqrt{AV_o(e_0) + V_c(\hat{w}_0)}.$$

Setting

$$M = \left(\frac{1}{\sqrt{A\epsilon}} + \frac{\|\mathcal{P}_c\|_{\mathcal{L}}}{\sqrt{\epsilon}}\right)\sqrt{AV_o(e_0) + V_c(\hat{w}_0)}$$

completes the proof. □

7.2.1 Numerical Results. To illustrate the effectiveness of the output feedback controller synthesis, we construct exponentially stabilizing boundary controllers for the PDEs considered in Chapter 6. We consider the following two parabolic PDEs:

$$w_t(x,t) = w_{xx}(x,t) + \lambda w(x,t), \text{ and} \tag{7.57}$$

$$w_t(x,t) = \left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t)$$
$$+ \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(x,t), \tag{7.58}$$

where λ is a scalar which may be chosen freely. We consider the following boundary conditions and outputs (y(t)) for these two equations:

$$\text{Dirichlet:} = w(0) = 0, \quad w(1) = u(t), \quad y(t) = w_x(1), \tag{7.59}$$
$$\text{Neumann:} = w_x(0) = 0, \quad w_x(1) = u(t), \quad y(t) = w(1), \tag{7.60}$$
$$\text{Mixed:} = w(0) = 0, \quad w_x(1) = u(t), \quad y(t) = w(1), \tag{7.61}$$
$$\text{Robin:} = w(0) + w_x(0) = 0, \quad w(1) + w_x(1) = u(t), \quad y(t) = w(1). \tag{7.62}$$

We apply Theorem 7.5 to these PDEs for different degrees of polynomial representation for parameter values $\epsilon = \delta = \delta_c = \delta_o = 0.001$. Table 7.1 and Figure 7.2 illustrate the maximum λ as a function of $d_1 = d_2 = d$ for which we can construct an exponentially stabilizing output feedback controller for Equation (7.57) using the analysis presented in Theorem 7.5.

Similarly Table 7.2 and Figure 7.3 illustrate the maximum λ for which we can construct an exponentially stabilizing output feedback controller for Equation (7.58) using the analysis presented in Theorem 7.5.

Similar to state feedback controller synthesis, from these results it is obvious that the conjecture that the proposed methodology can synthesize output feedback controllers for any controllable and observable class of PDE that we consider still holds. Additionally, the conditions of Theorem 7.5 are quite similar to the conditions of Theorem 6.4. Therefore, we infer that the inclusion of the integral kernels K_1, K_2, L_1 and L_2 (in Thm. 7.5) is vital.

Finally, we provide a numerical simulation of Equation (7.58) for $\lambda = 35$ and mixed boundary conditions while being acted upon the output feedback controller

Table 7.1. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 7.5 are feasible, thereby implying the existence of an exponentially stabilizing output feedback controller for Equation (7.57).

Boundary Conditions	$d = 6$	7	8	9	10	11
Dirichlet						
$w(0) = 0, w(1) = u(t)$	$\lambda = 10.3767$	14.3982	17.7643	22.8645	23.3093	27.1179
Neumann						
$w_x(0) = 0, w_x(1) = u(t)$	10.0739	13.1227	14.8163	17.1814	21.8781	21.8781
Mixed						
$w(0) = 0, w_x(1) = u(t)$	10.3767	14.3982	17.7643	22.8645	23.3093	27.1179
Robin						
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	9.1171	12.0911	14.9445	16.6565	18.7748	18.7748

Table 7.2. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 7.5 are feasible, thereby implying the existence of an exponentially stabilizing output feedback controller for Equation (7.58).

Boundary Conditions	$d = 4$	5	6	7	8
Dirichlet					
$w(0) = 0, w(1) = u(t)$	$\lambda = 18.3090$	36.0199	38.0478	40.5930	44.5219
Neumann					
$w_x(0) = 0, w_x(1) = u(t)$	15.8531	29.8492	32.4059	32.4059	34.1584
Mixed					
$w(0) = 0, w_x(1) = u(t)$	18.3090	36.0199	38.0478	40.5930	44.5219
Robin					
$w(0) + w_x(0) = 0, w(1) + w_x(1) = u(t)$	12.7869	24.7589	27.5421	27.9083	29.4762

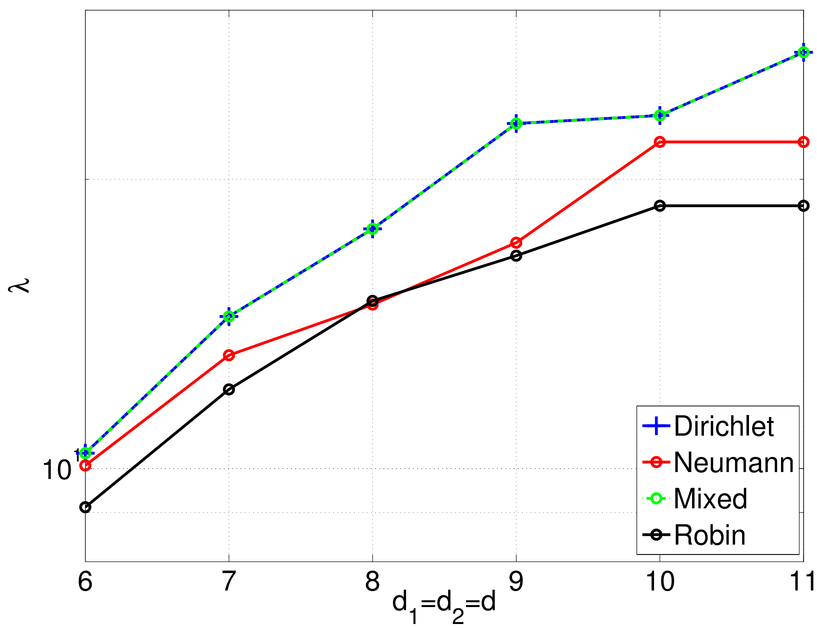


Figure 7.2. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 7.5 are feasible, thereby implying the existence of an exponentially stabilizing output feedback controller for Equation (7.57).

designed using Theorem 7.5. Figure 7.4 shows the response of the closed loop system with an initial condition

$$e^{-\frac{-(x-0.3)^2}{2(0.07)^2}} - e^{-\frac{-(x-0.7)^2}{2(0.07)^2}}.$$

Figure 7.5 shows the control input evolution.

Figure 7.6 shows the evolution of the observer state initialized by a zero initial condition.

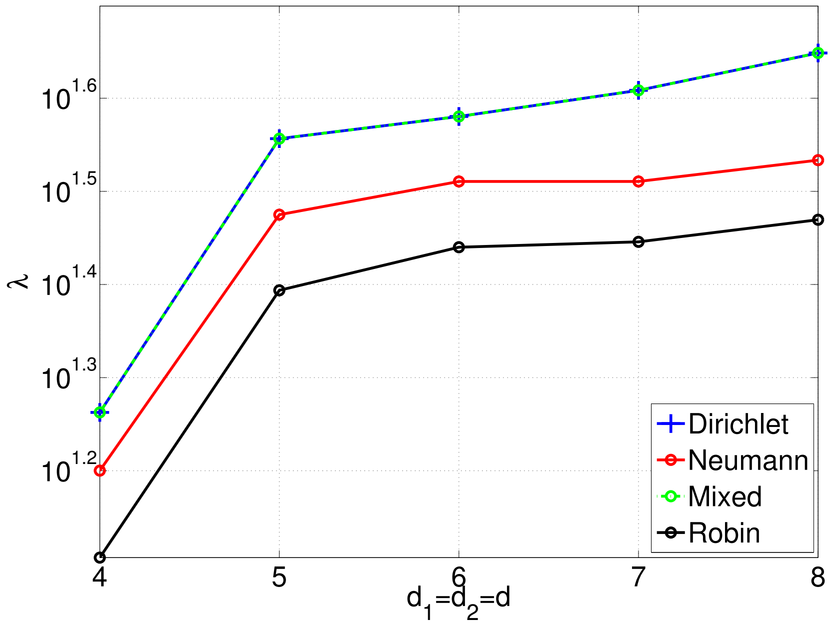


Figure 7.3. Maximum λ as a function of polynomial degree $d_1 = d_2 = d$ for which the conditions of Theorem 7.5 are feasible, thereby implying the existence of an exponentially stabilizing output feedback controller for Equation (7.58).

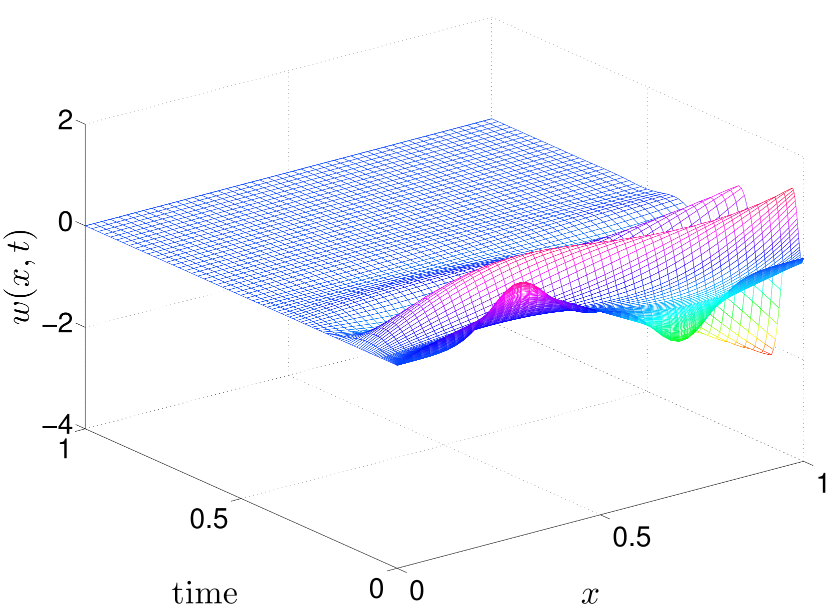


Figure 7.4. Closed loop state evolution of Equation (7.58) for $\lambda = 35$ and mixed boundary conditions .

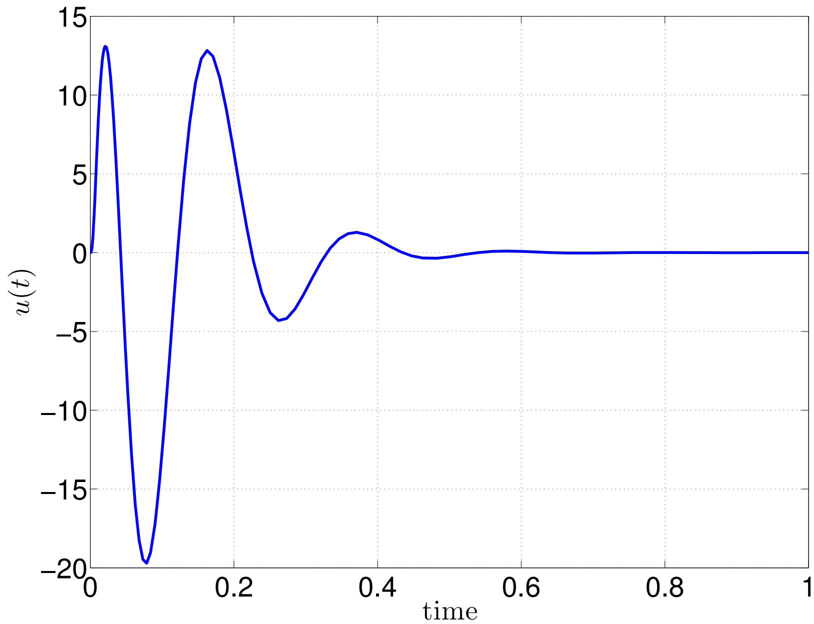


Figure 7.5. Control effort evolution of Equation (7.58) for $\lambda = 35$ and mixed boundary conditions .

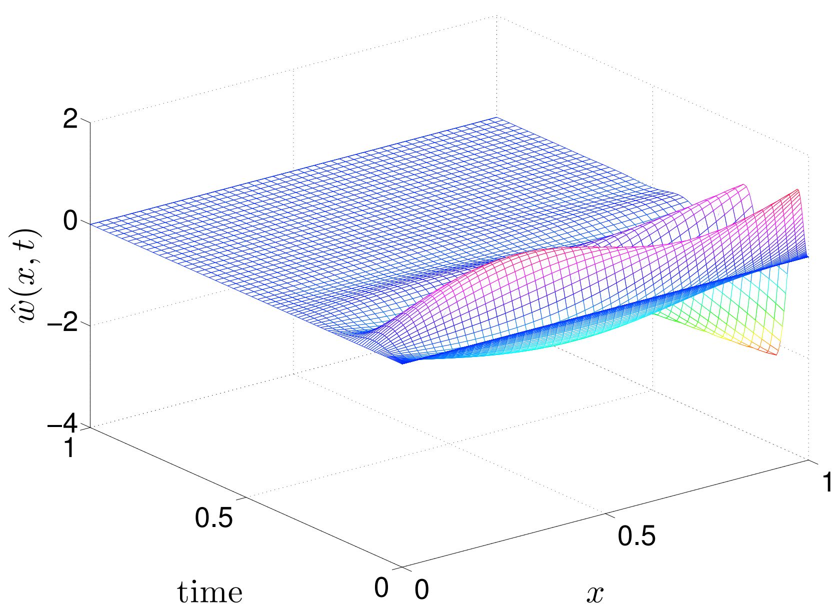


Figure 7.6. Observer state evolution.

CHAPTER 8

CONTROL AND VERIFICATION OF THE SAFETY FACTOR PROFILE IN TOKAMAKS

The instabilities in a tokamak plasma described by the *Magneto-Hydrodynamic-Dynamic* (MHD) models are known as *MHD instabilities*. MHD instabilities arise due to current gradients and pressure gradients interacting with the magnetic field line curvature [6].

A common heuristic for setting operating conditions that avoid MHD instabilities is the *safety factor profile*, or the q-profile [76]. Additionally, in [86], it has been shown that the safety factor profile is important in triggering Internal Transport Barriers (ITBs) which significantly improve energy confinement. The q-profile the the magnetic filed line pitch, that is, the number of revolutions a magnetic field line makes in the poloidal field while traversing a complete revolution in the toroidal plane. Recall the definition of the q-profile, presented in Equation (4.5),

$$q(x,t) = -\frac{B_{\phi_0} a^2 x}{Z(x,t)}, \tag{8.1}$$

where[3]

$$\begin{aligned}
B_{\phi_0} &= \text{toroidal magnetic field at the plasma center,} \\
a &= \text{loation of the last close magnetic surface,} \\
x &= \text{normalized spatial variable,} \\
t &= \text{temporal varable,} \\
Z(x,t) = \psi_x(x,t) &= \text{gradient of the poloidal magnetic flux, and} \\
\psi(x,t) &= \text{poloidal magnetic flux.}
\end{aligned}$$

From Equation (8.1), it is evident that to control the q-profile, we may control the gradient of the poloidal magnetic flux Z.

---

[3]Refer to Table 4.1 for tokamak variable definitions.

8.1 Simplified Model of the Gradient of Poloidal Flux

Recall the evolution equation of Z presented in Chapter 4 obtained by neglecting the diamagnetic effect and applying cylindrical approximation as

$$\frac{\partial Z}{\partial t}(x,t) = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x,t)}{x}\frac{\partial}{\partial x}\left(xZ(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)j_{ni}(x,t)\right), \tag{8.2}$$

with boundary conditions

$$Z(0,t) = 0 \text{ and } Z(1,t) = -R_0\mu_0 I_p(t)/2\pi, \tag{8.3}$$

where

$$\begin{aligned}
\eta_{\|} &= \text{parallel resistivity,}\\
j_{ni} &= \text{non-inductive effective current density,}\\
I_p &= \text{total plasma current,}\\
R_0 &= \text{location of magnetic center, and}\\
\mu_0 &= \text{permeability of free space.}
\end{aligned}$$

For this model, we consider the plasma resistivity $\eta_{\|}(x,t)$ to be static, thus $\eta_{\|}(x,t) = \eta_{\|}(x)$. Additionally, the averaged value of the bootsrap current density $j_{bs}(x,t) = \bar{j}_{bs}(x)$ is considered. For the external non-inductive current density source j_{eni}, we consider only the *Lower Hybrid Current Density* (LHCD) source j_{lh}. Finally, the plasma current I_p is considered to be constant. Thus, since, $j_{ni}(x,t) = j_{bs}(x,t) + j_{eni}(x,t)$, we obtain

$$j_{ni}(x,t) = \bar{j}_{bs}(x) + j_{lh}(x,t).$$

Substituting into Equation (8.2) and using the steady plasma resistivity $\eta_{\|}(x)$ and a constant I_p, we obtain

$$\frac{\partial Z}{\partial t}(x,t) = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(xZ(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x)\left[\bar{j}_{bs}(x) + j_{lh}(x,t)\right]\right), \tag{8.4}$$

with boundary conditions

$$Z(0,t) = 0 \text{ and } Z(1,t) = -R_0\mu_0 I_p/2\pi. \tag{8.5}$$

Suppose we want to regulate $q(x,t)$ to a desired steady state $q_{ref}(x)$. Let $Z_{ref}(x)$ be the associated gradient of the poloidal magnetic flux obtained using Equation (8.1). Then, since $Z_{ref}(x)$ satisfies Equations (8.4)-(8.5), we obtain

$$\frac{\partial Z_{ref}}{\partial t}(x) = 0 = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(xZ_{ref}(x)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x)\bar{j}_{bs}(x)\right), \tag{8.6}$$

with boundary conditions

$$Z_{ref}(0) = 0 \text{ and } Z_{ref}(1) = -R_0\mu_0 I_p/2\pi. \tag{8.7}$$

Subtracting Equations (8.6)-(8.7) from Equations (8.4)-(8.5) produces

$$\frac{\partial \hat{Z}}{\partial t}(x,t) = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(x\hat{Z}(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x)j_{lh}(x,t)\right), \tag{8.8}$$

with boundary conditions

$$\hat{Z}(0,t) = 0 \text{ and } \hat{Z}(1,t) = 0, \tag{8.9}$$

where

$$\hat{Z}(x,t) = Z(x,t) - Z_{ref}(x) \tag{8.10}$$

is the *error variable* which must be regulated to zero.

8.1.1 Uniqueness and Existence of Solutions. To regulate the error variable $\hat{Z}$ to zero, we will be constructing state feedback controllers of the form

$$j_{lh}(x,t) = K_1(x)\hat{Z}(x,t) + \frac{\partial}{\partial x}\left(K_2(x)\hat{Z}(x,t)\right), \tag{8.11}$$

where K_1 and K_2 are rational functions.

To establish the uniqueness and existence of solutions for Equations (8.8)-(8.9) with j_{lh} given in Equation (8.11), we will follow the procedure presented in Section 5.1. We begin by placing the following assumption.

Assumption 8.1. *The functions*

$$\frac{\eta_{\|}(x)}{x} + \eta_{\|,x}(x) \quad \textit{and} \quad \frac{x\eta_{\|,x}(x) - \eta_{\|}(x)}{x^2}$$

are continuous for $x \in [0, 1]$.

Lemma 8.2. *Suppose there exists a rational function* K_2 *such that*

$$\eta_{\|}(x)\left(\frac{1}{\mu_0 a^2} + R_0 K_2(x)\right) > 0, \quad x \in [0, 1].$$

Then, for any initial condition $\hat{Z}_0 \in \mathcal{D}_T$*, where*

$$\mathcal{D}_T = \{y \in H^2(0, 1) : \quad y(0) = y(1) = 0, \} \tag{8.12}$$

there exists a classical solution $\hat{Z}(\cdot, t) \in \mathcal{D}_T$*,* $t > 0$*, for Equations* (8.8)-(8.9) *with control given in Equation* (8.11) *with any rational function* K_1.

Similarly, for any initial condition $\hat{Z}_0 \in L_2(0, 1)$*, there exists a weak solution* $\hat{Z}(\cdot, t) \in L_2(0, 1)$*,* $t > 0$.

Proof. By substituting Equation (8.11) into Equation (8.8), we obtain

$$\frac{\partial \hat{Z}}{\partial t}(x, t) = a(x)\hat{Z}_{xx}(x, t) + b(x)\hat{Z}_x(x, t) + c(x)\hat{Z}(x, t), \tag{8.13}$$

with boundary conditions

$$\hat{Z}(0, t) = 0 \text{ and } \hat{Z}(1, t) = 0, \tag{8.14}$$

where

$$a(x) = \eta_{\|}(x)\left(\frac{1}{\mu_0 a^2} + R_0 K_2(x)\right),$$

$$b(x) = \frac{1}{\mu_0 a^2}\left(\frac{\eta_\|(x)}{x} + \eta_{\|,x}(x)\right) + R_0\left(\eta_\|(x)\left(K_1(x) + 2K_{2,x}(x)\right) + \eta_{\|,x}(x)K_2(x)\right),$$
$$c(x) = \frac{1}{\mu_0 a^2}\left(\frac{x\eta_{\|,x}(x) - \eta_\|(x)}{x^2}\right) + R_0\eta_\|(x)\left(K_{1,x}(x) + K_{2,xx}(x)\right)$$
$$+ R_0\eta_{\|,x}(x)\left(K_1(x) + K_{2,x}(x)\right).$$

For Equations (8.13)-(8.14), we define the following first order differential form

$$\dot{\hat{\mathbf{Z}}}(t) = \mathcal{A}_T\hat{\mathbf{Z}}(t), \tag{8.15}$$

where the operator $\mathcal{A}_T : H^2(0,1) \to L_2(0,1)$ is defined as

$$\left(\mathcal{A}_T y\right)(x) = a(x)y_{xx}(x) + b(x)y_x(x) + c(x)y(x), \quad y \in H^2(0,1). \tag{8.16}$$

From the theorem statement, $a(x) > 0$ for all $x \in [0,1]$. Moreover, from Assumption 8.1, the functions $b(x)$ and $c(x)$ are continuous. Thus, if we define

$$p(x) = e^{\int_0^x \frac{b(\xi)}{a(\xi)}d\xi}, \quad q(x) = -c(x)\frac{p(x)}{a(x)}, \quad \sigma(x) = \frac{p(x)}{a(x)},$$

it follows that, for any $y \in \mathcal{D}_T$,

$$-\mathcal{A}_T y = \frac{1}{\sigma(x)}\mathcal{S}y,$$

where $\mathcal{S}$ is the Sturm-Liouville operator defined as

$$\left(\mathcal{S}y\right)(x) = -\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right) + q(x)y(x), \quad y \in \mathcal{D}_T.$$

Therefore, similar to the analysis presented in Lemma 5.4, it can be established that the pair $(\mathcal{A}_T, \mathcal{D}_T)$ generates a C_0-semigroup $S(t)$ on $L_2(0,1)$. Thus, from Theorem A.3, for any initial condition $\hat{Z}_0 \in \mathcal{D}_T$, Equations (8.13)-(8.14) have a classical solution given by

$$\hat{Z}(x,t) = \left(S(t)\hat{Z}_0\right)(x). \tag{8.17}$$

From Corollary A.4, for any $\hat{Z}_0 \in L_2(0,1)$, (8.17) is the unique weak solution of (8.13)-(8.14). $\square$

8.2 Control Design

As explained before, we wish to design control j_{lh} of the form presented in Equation (8.11) such that $Z \to Z_{ref}$. As in previous chapters, we will use sum-of-squares polynomials.

We present the following theorem.

Theorem 8.3. *Suppose there exist polynomials $M(x)$, $Z_1(x)$ and $Z_2(x)$ and scalars ϵ, α such that, for all $x \in [0,1]$,*

$$\begin{aligned} M(x) \geq& \epsilon, \\ \frac{1}{\mu_0 a^2}\left(\mathcal{B}_1 M\right)(x) + \left(\mathcal{B}_2 Z_1\right)(x) + \left(\mathcal{B}_3 Z_2\right)(x) + \alpha f(x) M(x) <& 0, \\ \left(\mathcal{C}\left(\frac{1}{\mu_0 a^2} M + Z_2\right)\right)(x) <& 0, \end{aligned}$$

where $\mathcal{B}_i$, $i \in \{1,2,3\}$, and $\mathcal{C}$ are defined as

$$\begin{aligned} \left(\mathcal{B}_1 y\right)(x) =& \left(-f_x(x)\frac{\eta_{\|}(x)}{x} + \frac{1}{2}\frac{d}{dx}\left[f(x)\frac{\eta_{\|}(x)}{x} + f_x(x)\eta_{\|}(x)\right]\right) y(x) \\ &+ \frac{1}{2}\left(f_x(x)\eta_{\|}(x) + f(x)\frac{\eta_{\|}(x)}{x} + \frac{d}{dx}\left[f(x)\eta_{\|}(x)\right]\right)\frac{dy(x)}{dx} \\ &+ \frac{1}{2} f(x)\eta_{\|}(x)\frac{d^2 y(x)}{dx^2}, \quad y \in H^2(0,1), \\ \left(\mathcal{B}_2 y\right)(x) =& \frac{1}{2} f_x(x) y(x) - \frac{1}{2} f(x)\frac{dy(x)}{dx}, \quad y \in H^1(0,1), \\ \left(\mathcal{B}_3 y\right)(x) =& \frac{1}{2}\frac{d}{dx}(f_x(x)\eta_{\|}(x)) y(x) \\ &+ \frac{1}{2}\left(-f_x(x)\eta_{\|}(x) + \frac{d}{dx}\left(f(x)\eta_{\|}(x)\right)\right)\frac{dy(x)}{dx} \\ &+ \frac{1}{2} f(x)\eta_{\|}(x)\frac{d^2 y(x)}{dx^2}, \quad y \in H^2(0,1), \\ \left(\mathcal{C} y\right)(x) =& -\eta_{\|}(x) y(x), \quad y \in L_2(0,1), \\ f(x) =& x^2(1-x). \end{aligned}$$

Let

$$K_1(x) = R_0^{-1}\eta_{\|}(x)^{-1} M(x)^{-1} Z_1(x), \quad K_2(x) = R_0^{-1} M(x)^{-1} Z_2(x).$$

Then, with

$$j_{lh}(x,t) = K_1(x)\hat{Z}(x,t) + \frac{\partial}{\partial x}\left(K_2(x)\hat{Z}(x,t)\right),$$

for any initial condition $Z_0 \in \mathcal{D}_T(L_2(0,1))$ *and a desired reference profile* $Z_{ref} \in \mathcal{D}_T(L_2(0,1))$*, there exists a scalar* $\kappa \geq 0$ *such that*

$$\|Z(\cdot,t) - Z_{ref}(\cdot)\|_{L_2^f(0,1)} \leq \kappa e^{-\alpha t}, \quad t > 0,$$

where, for any $y \in L_2(0,1)$*,*

$$\|y\|_{L_2^f(0,1)} = \left(\int_0^1 f(x)y(x)^2 dx\right)^{\frac{1}{2}}.$$

Proof. We begin by recalling the evolution equation for $\hat{Z} = Z - Z_{ref}$ presented in Equation (8.8)-(8.9) as

$$\frac{\partial \hat{Z}}{\partial t}(x,t) = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(x\hat{Z}(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x)j_{lh}(x,t)\right), \tag{8.18}$$

with boundary conditions

$$\hat{Z}(0,t) = 0 \text{ and } \hat{Z}(1,t) = 0. \tag{8.19}$$

From the theorem statement, for all $x \in [0,1]$,

$$\left(\mathcal{C}\left(\frac{1}{\mu_0 a^2}M + Z_2\right)\right)(x) < 0.$$

Using the definition of $\mathcal{C}$ and $K_2(x)$, we obtain that

$$-M(x)\eta_{\|}(x)\left(\frac{1}{\mu_0 a^2} + R_0 K_2(x)\right) < 0.$$

Since $M(x) > 0$, we conclude that, for all $x \in [0,1]$,

$$\eta_{\|}(x)\left(\frac{1}{\mu_0 a^2} + R_0 K_2(x)\right) > 0.$$

Therefore, from Lemma 8.2, if $Z_0, Z_{ref} \in \mathcal{D}_T(L_2(0,1))$, and consequently, $\hat{Z} \in \mathcal{D}_T(L_2(0,1))$, Equations (8.18)-(8.19) have a classical (weak) solution.

With the uniqueness and existence of solutions to Equations (8.18)-(8.19) established, let us define the following Lyapunov function

$$V(\hat{Z}(\cdot,t)) = \int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)^2 dx.$$

Taking the derivative along the trajectories of (8.18)-(8.19),

$$\begin{aligned}\dot{V}(\hat{Z}(\cdot,t)) =& 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\hat{Z}_t(x,t)dx\\ =& \frac{2}{\mu_0 a^2}\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(x\hat{Z}(x,t)\right)\right)dx\\ &+ 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\left[R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x)j_{lh}(x,t)\right)\right]dx\end{aligned}$$

Substituting in

$$j_{lh}(x,t) = K_1(x)\hat{Z}(x,t) + \frac{\partial}{\partial x}\left(K_2(x)\hat{Z}(x,t)\right)$$

produces

$$\begin{aligned}\dot{V}(\hat{Z}(\cdot,t)) =& \frac{2}{\mu_0 a^2}\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(x\hat{Z}(x,t)\right)\right)dx\\ &+ 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left(R_0\eta_{\|}(x)K_1(x)\hat{Z}(x,t)\right)dx\\ &+ 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left[\eta_{\|}(x)\frac{\partial}{\partial x}\left(R_0K_2(x)\hat{Z}(x,t)\right)\right]dx.\end{aligned}$$

Since,

$$K_1(x) = R_0^{-1}\eta_{\|}(x)^{-1}M(x)^{-1}Z_1(x), \quad K_2(x) = R_0^{-1}M(x)^{-1}Z_2(x),$$

we have that

$$\begin{aligned}\dot{V}(\hat{Z}(\cdot,t)) =& \frac{2}{\mu_0 a^2}\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(x\hat{Z}(x,t)\right)\right)dx\\ &+ 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left(Z_1(x)M(x)^{-1}\hat{Z}(x,t)\right)dx\\ &+ 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left[\eta_{\|}(x)\frac{\partial}{\partial x}\left(Z_2(x)M(x)^{-1}\hat{Z}(x,t)\right)\right]dx.\end{aligned}$$

We can write

$$\dot{V}(\hat{Z}(\cdot,t))$$

$$= \frac{2}{\mu_0 a^2} \int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(xM(x)M(x)^{-1}\hat{Z}(x,t)\right)\right)dx$$
$$+ 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left(Z_1(x)M(x)^{-1}\hat{Z}(x,t)\right)dx$$
$$+ 2\int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)\frac{\partial}{\partial x}\left[\eta_{\|}(x)\frac{\partial}{\partial x}\left(Z_2(x)M(x)^{-1}\hat{Z}(x,t)\right)\right]dx.$$

If we define

$$Y(x,t) = M(x)^{-1}\hat{Z}(x,t),$$

we get

$$\begin{aligned}\dot{V}(\hat{Z}(\cdot,t)) =& \frac{2}{\mu_0 a^2}\int_0^1 f(x)Y(x,t)\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(xM(x)Y(x,t)\right)\right)dx\\ &+ 2\int_0^1 f(x)Y(x,t)\frac{\partial}{\partial x}\left(Z_1(x)Y(x,t)\right)dx\\ &+ 2\int_0^1 f(x)Y(x,t)\frac{\partial}{\partial x}\left[\eta_{\|}(x)\frac{\partial}{\partial x}\left(Z_2(x)Y(x,t)\right)\right]dx.\end{aligned}$$

Thus, we can write

$$\dot{V}(\hat{Z}(\cdot,t)) = \frac{2}{\mu_0 a^2}\dot{V}_1(\hat{Z}(\cdot,t)) + 2\dot{V}_2(\hat{Z}(\cdot,t)) + 2\dot{V}_3(\hat{Z}(\cdot,t)), \tag{8.20}$$

where

$$\begin{aligned}\dot{V}_1(\hat{Z}(\cdot,t)) &= \int_0^1 f(x)Y(x,t)\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x)}{x}\frac{\partial}{\partial x}\left(xM(x)Y(x,t)\right)\right)dx,\\ \dot{V}_2(\hat{Z}(\cdot,t)) &= \int_0^1 f(x)Y(x,t)\frac{\partial}{\partial x}\left(Z_1(x)Y(x,t)\right)dx,\\ \dot{V}_3(\hat{Z}(\cdot,t)) &= \int_0^1 f(x)Y(x,t)\frac{\partial}{\partial x}\left[\eta_{\|}(x)\frac{\partial}{\partial x}\left(Z_2(x)Y(x,t)\right)\right]dx.\end{aligned}$$

Before simplifying these terms using integration by parts, we would like to comment that since $Y(x,t) = M(x)^{-1}\hat{Z}(x,t)$, from (8.19), we obtain that

$$Y(0,t) = 0 \text{ and } Y(1,t) = 0. \tag{8.21}$$

Applying integration by parts twice and using (8.21) produces

$$\dot{V}_1(\hat{Z}(\cdot,t)) = \int_0^1 Y(x,t)^2\left(\mathcal{B}_1 M\right)(x)dx + \int_0^1 Y_x(x,t)^2 f(x)\left(\mathcal{C}M\right)(x)dx. \tag{8.22}$$

Applying integration by parts once,

$$\dot{V}_2(\hat{Z}(\cdot,t)) = \int_0^1 Y(x,t)^2 \left(\mathcal{B}_2 Z_1\right)(x)dx. \tag{8.23}$$

Finally, applying integration by parts twice produces

$$\dot{V}_3(\hat{Z}(\cdot,t)) = \int_0^1 Y(x,t)^2 \left(\mathcal{B}_3 Z_2\right)(x)dx + \int_0^1 Y_x(x,t)^2 f(x) \left(\mathcal{C} Z_2\right)(x)dx.. \tag{8.24}$$

Substituting Equations (8.22)-(8.24) into (8.20) produces

$$\begin{aligned}\dot{V}(\hat{Z}(\cdot,t)) =& 2\int_0^1 Y(x,t)^2 \left(\frac{1}{\mu_0 a^2}\left(\mathcal{B}_1 M\right)(x) + \left(\mathcal{B}_2 Z_1\right)(x)\left(\mathcal{B}_3 Z_2\right)(x)\right)dx \\ &+ 2\int_0^1 Y_x(x,t)^2 \left(f(x)\mathcal{C}\left(\frac{1}{\mu_0 a^2}M + Z_2\right)(x)\right)dx.\end{aligned}$$

Now

$$V(\hat{Z}(\cdot,t)) = \int_0^1 f(x)M(x)^{-1}\hat{Z}(x,t)^2 dx = \int_0^1 f(x)M(x)Y(x,t)^2 dx.$$

Thus

$$\begin{aligned}&\dot{V}(\hat{Z}(\cdot,t)) + 2\alpha V(\hat{Z}(\cdot,t)) \\ &= 2\int_0^1 Y(x,t)^2 \left(\frac{1}{\mu_0 a^2}\left(\mathcal{B}_1 M\right)(x) + \left(\mathcal{B}_2 Z_1\right)(x)\left(\mathcal{B}_3 Z_2\right)(x) + \alpha f(x)M(x)\right)dx \\ &\quad + 2\int_0^1 Y_x(x,t)^2 \left(f(x)\mathcal{C}\left(\frac{1}{\mu_0 a^2}M + Z_2\right)(x)\right)dx. \end{aligned} \tag{8.25}$$

Since, from the theorem statement, for all $x \in [0,1]$,

$$\begin{aligned}\frac{1}{\mu_0 a^2}\left(\mathcal{B}_1 M\right)(x) + \left(\mathcal{B}_2 Z_1\right)(x)\left(\mathcal{B}_3 Z_2\right)(x) + \alpha f(x)M(x) <& 0, \\ \mathcal{C}\left(\frac{1}{\mu_0 a^2}M + Z_2\right)(x) <& 0,\end{aligned}$$

and $f(x) \geq 0$, from Equation (8.25)

$$\dot{V}(\hat{Z}(\cdot,t)) \leq -2\alpha V(\hat{Z}(\cdot,t)).$$

Thus, integrating in time

$$V(\hat{Z}(\cdot,t)) \leq e^{-2\alpha t}V(\hat{Z}_0) = e^{-2\alpha t}V(Z_0 - Z_{ref}). \tag{8.26}$$

Using the fact that $M(x) \geq \epsilon > 0$, thus

$$\|Z(\cdot,t) - Z_{ref}(\cdot)\|^2_{L_2^f(0,1)} \leq \frac{1}{\inf_{x\in[0,1]} M(x)} e^{-2\alpha t} V(Z_0 - Z_{ref}).$$

Taking the square root and setting

$$\kappa = \sqrt{\frac{V(Z_0 - Z_{ref})}{\inf_{x\in[0,1]} M(x)}},$$

completes the proof. □

8.3 Numerical Simulation

We test the conditions of Theorem 8.3 using SOSTOOLS. Once we obtain polynomials $M(x)$, $Z_1(x)$ and $Z_2(x)$, and designed a controller, we would like to simulate the dynamics under realistic operating conditions. For this we discretize the error dynamics given by Equations (8.8)-(8.9) with control given by Equation (8.11). However, unlike Chapters 5-7, a simple finite-difference scheme cannot be applied to disctretize the system dynamics. This is due to the fact that the coefficients of the PDE in question have a singularity at $x = 0$. This problem may be overcome by modifying the finite difference scheme as explained in [87].

For the purpose of simulation, the following values are taken from the data of the Tore Supra tokamak: $I_p = 0.6MA$ and $B_{\phi_0} = 1.9T$, where I_p is the plasma current and B_{ϕ_0} is the toroidal magnetic field at the plasma center.

Given a q_{ref}-profile, the corresponding Z_{ref}-profile can be computed using (8.1), where $a = 0.38$ m for Tore Supra. The boundary values for Z are calculated using the magnetic center location, which is $R_0 = 2.38$ m and (8.5) to get

$$Z(0,t) = 0 \text{ and } Z(1,t) = -0.2851. \tag{8.27}$$

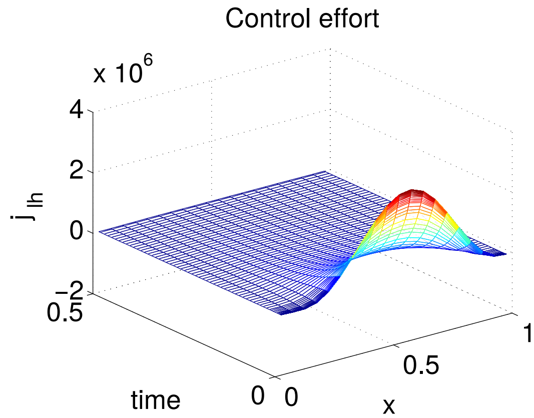
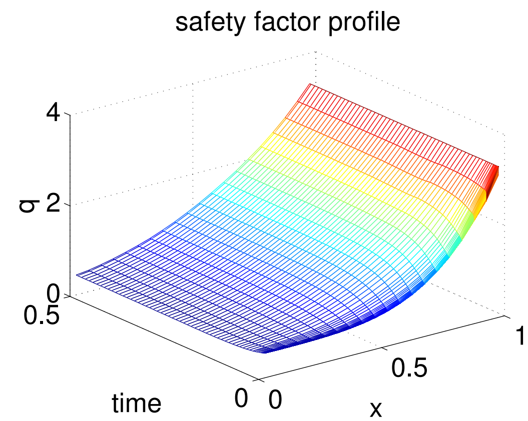
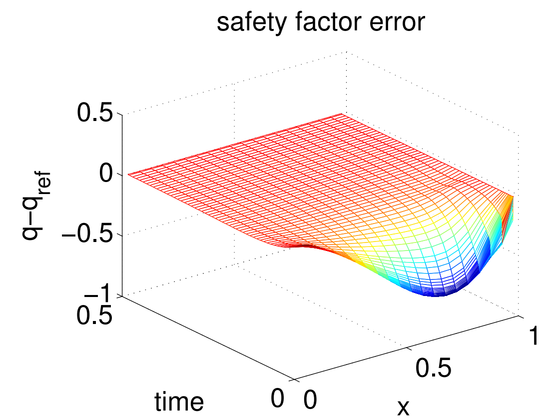


Figure 8.1. Control effort, $j_{lh}(x,t)$.

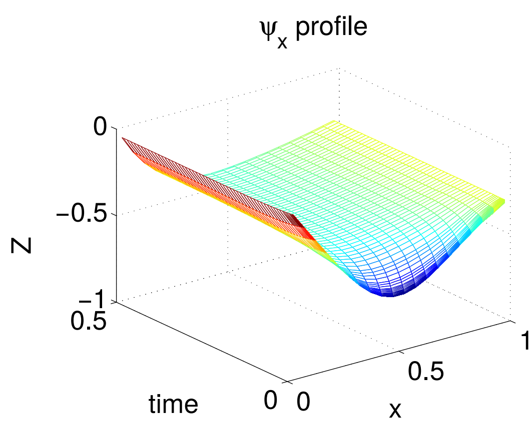
Even though we used steady-state $\eta_{\|}$ for controller synthesis, in order for a realistic controller simulation we use time-varying $\eta_{\|}$ data for shot TS 35109. Time evolution of the pertinent variables is presented in Figs. 8.1-8.2.



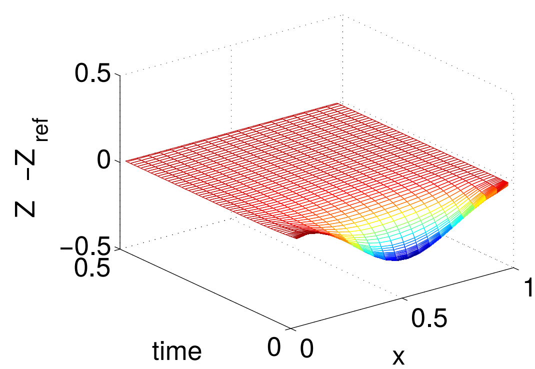
(a) Time evolution of the safety factor profile or the q-profile.



(b) Time evolution of the q-profile Error, $q(x,t) - q_{ref}(x)$.



(c) Time evolution of Z-profile corresponding to the q-profile in Fig. 8.2(a).



(d) Z-profile error, $\hat{Z} = Z - Z_{ref}$. Here Z_{ref} is obtained from the reference q-profile, q_{ref}.

Figure 8.2. Time evolution of safety-factor and Z profiles and their corresponding error profiles

CHAPTER 9

MAXIMIZATION OF BOOTSTRAP CURRENT DENSITY IN TOKAMAKS

In order to contain plasma, a tokamak uses a helical magnetic field which is generated due to the superposition of toroidal and poloidal magnetic fields. The toroidal magnetic field is generated using powerful external electromagnets, whereas, the poloidal magnetic field is generated by the plasma current I_p. A major fraction of I_p comes from the current induced by the central ohmic coil using transformer effect. Other sources of I_p are the external non-inductive sources of *Lower Hybrid Current Density* (LHCD) and *Electron Cyclotron Current Density* (ECCD). The total current provided provided by these sources accounts for a considerable portion of energy required for tokamak operation. Moreover, due to the current induced by the ohmic coil accounting for a large portion of I_p, a tokamak can only operate as a pulsed device.

An additional source of current is internally generated by particles trapped between isoflux surfaces (surfaces with constant magnetic flux). This current is referred to as the *bootstrap current* [6]. Thus, bootstrap current is an automatically generated source contributing to I_p. A brief explanation of the mechanism which leads to the generation of the bootstrap current is provided in Chapter 4. An increase in the bootstrap current would lead to a reduced dependence on the current generated by the ohmic coil induction and the LHCD and ECCD inputs. This reduced dependence on external current sources would also increase the pulse lengths for which the tokamak can operate. For example, the ultimate goal of the ITER project [88] is to demonstrate the steady state operation of tokamaks. A high value of bootstrap current has been identified as a crucial factor for steady state operation of tokamaks [89], [90].

From Equation (4.4), we have that the bootstrap current density can be expressed as a function of the electron and ion temperature and density profiles and the gradient of the poloidal magnetic flux $Z = \psi_x$ as

$$j_{bs}(x,t) = \frac{C(x,t)}{Z(x,t)}, \tag{9.1}$$

where[4]

$$\begin{aligned}
C(x,t) =& eR_0 \left((A_1 - A_2) n_e \frac{\partial T_e}{\partial x} + A_1 T_e \frac{\partial n_e}{\partial x} + A_1 (1-\alpha_i) n_i \frac{\partial T_i}{\partial x} + A_1 T_i \frac{\partial n_i}{\partial x} \right), \\
n_i(n_e) =& \text{ ion (electron) density profile,} \\
T_i(T_e) =& \text{ ion (electron) temperature profile,} \\
\alpha_i =& \text{ ion thermal speed,} \\
e =& \text{ electron charge,} \\
R_0 =& \text{ location of magnetic center, and} \\
A_1, A_2 =& \text{ functions of ratio of trapped to free particles.}
\end{aligned}$$

It is evident from Equation (9.1) that in order to maximize j_{bs}, the gradient of the poloidal magnetic flux Z may be minimized. In this chapter, we construct controllers which allow us to minimize the upper bound on the norm of Z.

9.1 Model of the Gradient of the Poloidal Flux

Recall the evolution equation of $Z = \psi_x$, ψ being the poloidal magnetic flux, presented in Chapter 4 obtained by neglecting the diamagnetic effect and applying cylindrical approximation as

$$\frac{\partial Z}{\partial t}(x,t) = \frac{1}{\mu_0 a^2} \frac{\partial}{\partial x} \left(\frac{\eta_{\|}(x,t)}{x} \frac{\partial}{\partial x} \left(x Z(x,t) \right) \right) + R_0 \frac{\partial}{\partial x} \left(\eta_{\|}(x,t) j_{ni}(x,t) \right), \tag{9.2}$$

with boundary conditions

$$Z(0,t) = 0 \text{ and } Z(1,t) = -R_0 \mu_0 I_p(t)/2\pi, \tag{9.3}$$

[4]Refer to Table 4.1 for tokamak variable definitions

where

$$\eta_{\|} = \text{parallel resistivity},$$

$$j_{ni} = \text{non-inductive effective current density},$$

$$I_p = \text{total plasma current, and}$$

$$\mu_0 = \text{permeability of free space.}$$

The non-inductive current density j_{ni} is a sum of the bootstrap current density j_{bs} and the external non-inductive current density j_{eni}. Moreover, as in Chapter 8, we will consider only the Lower Hybrid Current Density (LHCD) as j_{eni}. Thus

$$j_{ni} = j_{bs} + j_{lh}.$$

Hence, the model can be written as

$$\begin{aligned}\frac{\partial Z}{\partial t}(x,t) =& \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x,t)}{x}\frac{\partial}{\partial x}\left(xZ(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)j_{bs}(x,t)\right)\\ &+ R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)j_{lh}(x,t)\right).\end{aligned} \tag{9.4}$$

In our analysis, we will assume that

$$Z_x(1,t) = -Z(1,t). \tag{9.5}$$

This assumption is based on the observation that the total current density $j_T(x,t)$, defined in [67] as

$$j_T(x,t) = -\frac{xZ_x(1,t) + Z(x,t)}{\mu_0 R_0 a^2 x},$$

is weak at the plasma edge, however, we assume it to be zero.

Recall from Equation (9.1) that $j_{bs}(x,t) = C(x,t)/Z(x,t)$. As a result Equation (9.4) is implicitly nonlinear in Z. We address this problem by linearizing j_{bs} about a static operating point $\bar{Z}(x)$ to get

$$j_{bs}(x,t) = \frac{\bar{C}(x)}{\bar{Z}(x)} + u(x,t),$$

where $\bar{C}(x)$ corresponds to the static operating point $\bar{Z}(x)$ and

$$u(x,t) = \frac{\partial}{\partial Z} C|_{Z=\bar{Z}} \left(Z(x,t) - \bar{Z}(x) \right).$$

For our analysis, we take $\bar{C}(x)/\bar{Z}(x) = 0$. Numerical simulation results presented at the end of the chapter verify that this assumption does not have a significant effect on the controller performance. Thus

$$j_{bs}(x,t) = u(x,t).$$

Substituting into Equation (9.4) produces the evolution equation Z used for the controller synthesis and is given by

$$\begin{aligned} \frac{\partial Z}{\partial t}(x,t) =& \frac{1}{\mu_0 a^2} \frac{\partial}{\partial x} \left(\frac{\eta_{\|}(x,t)}{x} \frac{\partial}{\partial x} \left(x Z(x,t) \right) \right) + R_0 \frac{\partial}{\partial x} \left(\eta_{\|}(x,t) j_{lh}(x,t) \right) \\ &+ R_0 \frac{\partial}{\partial x} \left(\eta_{\|}(x,t) u(x,t) \right). \end{aligned} \tag{9.6}$$

with boundary conditions

$$Z(0,t) = 0 \text{ and } Z(1,t) = -R_0 \mu_0 I_p(t)/2\pi. \tag{9.7}$$

We will take the disturbance $u(x,t)$ to be the external input to the system and assume that $u \in L_2^{loc}([0,\infty], C^2(0,1)) \subset L_2^{loc}([0,\infty], L_2(0,1))$[5]. This also implies that for all $0 < T < \infty$, $u \in L_2([0,T], C^2(0,1)) \subset L_2([0,T], L_2(0,1))$. Unlike Chapters 5-8, where the coefficient of the PDEs involved were only spatially varying, the coefficients in Equation (9.6) are time-varying due to the presence of $\eta_{\|}(x,t)$. Thus, we can no longer apply the semigroup approach to prove the uniqueness and existence of solutions. Instead, we assume that for all initial conditions $Z_0 \in C^2[0,1]$ and all sufficiently smooth $\eta_{\|}$, there exists a unique solution $Z \in C^1([0,T], C^2(0,1))$ satisfying Equations (9.6)-(9.7). Refer to [33, Section 7.6] for the existence and uniqueness of

[5]Refer to Section 2.2 for the definitions of the function spaces

solutions to parabolic PDEs with time-varying coefficients. Improved regularity for zero boundary conditions has been proved in [91].

9.1.1 Control Input. The control input j_{lh} is shape constrained. The shape constraints are dependent on the operating conditions. Using the X-ray measurement from *Tore Supra* and empirical model of j_{lh} was developed in [47] and is presented in Chapter 4. This model uses a Gaussian deposition pattern with control authority over certain scaling parameters. In particular, we may use

$$j_{lh}(x,t) = v_{lh}(t)e^{-(\mu_{lh}(t)-x)^2/2\sigma_{lh}(t)}, \tag{9.8}$$

where we may control the amplitude v_{lh}, mean μ_{lh} and the variance σ_{lh} with the constraints that $v_{lh}(t) \in [0, 1.22 \text{ MA}]$, $\mu_{lh}(t) \in [0.14, 0.33]$, and $\sigma_{lh}(t) \in [0.016, 0.073]$, for all $t \geq 0$.

We will design control laws for these three input parameters using full-state feedback. Note that we choose the Gaussian parameters as the control input parameters and not the engineering parameters, namely the hybrid wave parallel refractive index $N_\parallel$ and the lower hybrid antenna power P_{lh}. In a tokamak, these parameters determine the Gaussian parameters. Hence, unlike the approach we have chosen, the mean, amplitude and variance of the control cannot vary independently.

9.2 A Boundedness Condition on the System Solution

We wish to synthesize control j_{lh} such that the norm of Z is minimized in the presence of the input u. We now present a result which shows that, for a bounded u, Z is bounded.

Lemma 9.1. *Consider the function*

$$V(Z(\cdot,t)) = \int_0^1 Z(x,t)f(x)M(x)^{-1}Z(x,t)dx,$$

where $f(x) = x^2$*,* $M(x) > 0$*, for all* $x \in [0,1]$*, and* Z *is the solution of Equations* (9.6)-(9.7) *with* $u \in L_2^{loc}([0,\infty], C^2(0,1))$*. Suppose that there exists a scalar* $\gamma > 0$ *such that*

$$\frac{dV(Z(\cdot,t))}{dt} = \dot{V}(Z(\cdot,t)) \leq \frac{1}{\gamma}\|u(\cdot,t)\|^2 - \gamma\|Z(\cdot,t)\|^2_{L_2^{M^{-2}}(0,1)},$$

for all $t \geq 0$*. Then*

$$\|Z\|^2_{L_2^{loc}([0,\infty],L_2^{M^{-2}}(0,1))} \leq \frac{1}{\gamma^2}\|u\|^2_{L_2^{loc}([0,\infty],L_2^{M^{-2}}(0,1))} + \frac{1}{\gamma}V(Z_0),$$

where $Z_0 \in C^2[0,1]$ *is the initial condition.*

Here,

$$L_2^{M^{-2}}(0,1) := \{g : (0,1) \to \mathbb{R} : \|g\|_{L_2^{M^{-2}}} = \left(\int_0^1 M(x)^{-2} g(x)^2 dx\right)^{\frac{1}{2}} < \infty\}.$$

Proof. Since $u \in L_2^{loc}([0,\infty], C^2(0,1))$, for any $0 < T < \infty$, we have that $u \in L_2([0,T], C^2(0,1))$. Thus, from our assumption, for any initial condition $Z_0 \in C^2[0,1]$, there exists a unique $Z \in C^1([0,T], C^2(0,1))$ satisfying Equations (9.6)-(9.7). Additionally

$$\frac{1}{2}\dot{V}(Z(\cdot,t)) = \int_0^1 Z(x,t) f(x) M(x)^{-1} \frac{\partial Z}{\partial t}(x,t).$$

Note that this is well defined as $\partial Z(x,t)/\partial t$ is given by (9.6) and $f(x)$ cancels out the singularity at $x = 0$ due to $1/x$.

Assume that the hypothesis of the Lemma holds. Integrating

$$\dot{V}(Z(\cdot,t)) \leq \frac{1}{\gamma}\|u(\cdot,t)\|^2 - \gamma\|Z(\cdot,t)\|^2_{L_2^{M^{-2}}(0,1)}$$

in time from 0 to an arbitrary $0 < T < \infty$,

$$\|Z\|^2_{L_2([0,T],L_2^{M^{-2}}(0,1))} \leq \frac{1}{\gamma^2}\|u\|^2_{L_2([0,T],L_2^{M^{-2}}(0,1))} + \frac{1}{\gamma}V(Z_0),$$

where we have used the fact that $Z(x,0) = Z_0(x)$.

Taking the limit $T \to \infty$ gives us

$$\|Z\|^2_{L_2^{loc}([0,\infty],L_2^{M-2}(0,1))} \leq \frac{1}{\gamma^2}\|u\|^2_{L_2^{loc}([0,\infty],L_2^{M-2}(0,1))} + \frac{1}{\gamma}V(Z_0).$$

This expression is well defined since $\|u\|^2_{L_2^{loc}([0,\infty],L_2^{M-2}(0,1))} < \infty$ and $V(Z_0)/\gamma$ is a constant.

□

9.3 Control Design

We now apply integration by parts to the condition in Lemma 9.1 to formulate our optimization problem which will allow us to synthesize controllers which minimize the upper bound $\frac{1}{\gamma}$ on Z. We assume that the plasma resistivity can be approximates, as given in [91]:

$$\eta_{\|}(x,t) = a(t)e^{\lambda(t)}x,$$

where, for all $t \geq 0$, $0 < \underline{a} \leq a(t) \leq \bar{a} < \infty$ and $0 < \underline{\lambda} \leq \lambda(t) \leq \bar{\lambda} < \infty$.

We present the following theorem.

Theorem 9.2. *Suppose that for a given scalar $\gamma > 0$ there exist polynomials $M(x)$ and $R(x)$ such that*

$$\begin{aligned} M(x) >&0, \;\; \text{for all } x \in [0,1], \\ \Omega(x,\lambda) + \Theta \leq&0, \;\; \text{for all } (x,\lambda) \in [0,1] \times [\underline{\lambda}, \bar{\lambda}], \\ 2A_4 + 2B_2 + A_2(1) \leq&0, \end{aligned}$$

where

$$\Omega(x,\lambda) = \begin{bmatrix} 2A_1(x) & 0 & -R_0\mu_0 a^2 f(x) \\ \star & A_0(x,\lambda) & -R_0\mu_0 a^2 f_x(x) \\ \star & \star & 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} 0 & 0 & 0 \\ \star & \frac{\mu_0 a^2 \gamma}{\underline{a}} & 0 \\ \star & \star & -\frac{\mu_0 a^2}{\bar{a}e^{\lambda}\gamma} \end{bmatrix},$$

$$\begin{aligned}
A_0(x,\lambda) =& 2A_3(x) - \lambda A_2(x) - A_{2,x}(x) + 2B_1(x,\lambda), \quad A_1(x) = -f(x)M(x),\\
A_2(x) =& -\bar{f}(x)M(x) - f(x)M_x(x) - f_x(x)M(x),\\
A_3(x) =& -2M(x) - f_x(x)M_x(x), \quad A_4 = M(1),\\
B_1(x) =& \frac{1}{2}\left(-f_x(x)R(x) + f(x)R_x(x) + \lambda f(x)R(x)\right), \quad B_2 = \frac{1}{2}R(1),\\
f(x) =& x^2 \text{ and } \bar{f}(x) = x.
\end{aligned}$$

Then if

$$j_{lh}(x,t) = \frac{K(x)}{R_0\mu_0 a^2}Z(x,t),$$

where $K(x) = M(x)^{-1}R(x)$*, then* Z *is bounded as follows:*

$$\|Z\|^2_{L_2^{loc}([0,\infty],L_2^{M^{-2}}(0,1))} \le \frac{1}{\gamma^2}\|u\|^2_{L_2^{loc}([0,\infty],L_2^{M^{-2}}(0,1))} + \frac{1}{\gamma}V(Z_0).$$

Proof. Suppose there exists a $\gamma > 0$ for which the hypotheses of the theorem hold true. Taking the time derivative of $V(Z(\cdot,t))$ defined in Lemma 9.1 produces

$$\begin{aligned}
\frac{1}{2}\dot{V}(Z(\cdot,t)) =& \int_0^1 Z(x,t)M(x)^{-1}f(x)\frac{\partial Z}{\partial t}(x,t)dx,\\
=& \dot{V}_1(Z(\cdot,t)) + \dot{V}_2(Z(\cdot,t)) + \dot{V}_3(Z(\cdot,t)),
\end{aligned}$$

where

$$\begin{aligned}
\dot{V}_1(Z(\cdot,t)) =& \frac{1}{\mu_0 a^2}\int_0^1 Z(x,t)M(x)^{-1}f(x)\frac{\partial}{\partial x}\left(\frac{\eta_\|(x,t)}{x}\frac{\partial}{\partial x}(xZ(x,t))\right)dx,\\
\dot{V}_2(Z(\cdot,t)) =& R_0\int_0^1 Z(x,t)M(x)^{-1}f(x)\frac{\partial}{\partial x}\left(\eta_\|(x,t)u(x,t)\right)dx,\\
\dot{V}_3(Z(\cdot,t)) =& R_0\int_0^1 Z(x,t)M(x)^{-1}f(x)\frac{\partial}{\partial x}\left(\eta_\|(x,t)j_{lh}(x,t)\right)dx.
\end{aligned}$$

If we define

$$Y(x,t) = M(x)^{-1}Z(x,t),$$

we obtain

$$\dot{V}_1(Z(\cdot,t)) = \frac{1}{\mu_0 a^2}\int_0^1 Y(x,t)f(x)\frac{\partial}{\partial x}\left(\frac{\eta_\|(x,t)}{x}\frac{\partial}{\partial x}(xM(x)Y(x,t))\right)dx,$$

$$\dot{V}_2(Z(\cdot,t)) = R_0 \int_0^1 Y(x,t)f(x)\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)u(x,t)\right)dx,$$
$$\dot{V}_3(Z(\cdot,t)) = R_0 \int_0^1 Y(x,t)f(x)\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)j_{lh}(x,t)\right)dx.$$

Applying integration by parts twice, we obtain

$$\begin{aligned}\dot{V}_1(Z(\cdot,t)) = &\int_0^1 \frac{\eta_{\|}(x,t)}{\mu_0 a^2}A_1(x)Y_x(x,t)^2dx \\ &+ \int_0^1 \frac{\eta_{\|}(x,t)}{\mu_0 a^2}\left(A_3(x) - \frac{1}{2}\lambda A_2(x) - \frac{1}{2}A_{2,x}(x)\right)Y(x,t)^2dx \\ &+ \frac{\eta_{\|}(1,t)}{\mu_0 a^2}\left(A_4 + \frac{1}{2}A_2(1)\right)Y(1,t)^2 + \frac{\eta_{\|}(1,t)}{\mu_0 a^2}Z_x(1,t)Y(1,t). \end{aligned} \tag{9.9}$$

Here we have used the fact that

$$\begin{aligned} &Z(x,t) = M(x)Y(x,t), \\ &\Rightarrow Z_x(x,t) = M_x(x)Y(x,t) + M(x)Y_x(x,t), \\ &\Rightarrow Z_x(1,t) = M_x(1)Y(1,t) + M(1)Y_x(1,t). \end{aligned}$$

Due to the assumption on the total current density on the boundary $j_T(1,t)$ and due to the linearization of j_{bs}, we obtain the boundary condition $u(1,t) = 0$. Thus, upon applying integration by parts once, we obtain

$$\dot{V}_2(Z(\cdot,t)) = -\int_0^1 R_0\eta_{\|}(x,t)\left(Y(x,t)f_x(x) + Y_x(x,t)f(x)\right)u(x,t)dx. \tag{9.10}$$

Using the feedback law $j_{lh}(x,t) = K(x)Z(x,t)/R_0\mu_0 a^2$, we get

$$\begin{aligned}\dot{V}_3(Z(\cdot,t)) =& \frac{1}{\mu_0 a^2}\int_0^1 Y(x,t)f(x)\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)K(x)Z(x,t)\right)dx \\ =& \frac{1}{\mu_0 a^2}\int_0^1 Y(x,t)f(x)\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)K(x)M(x)M(x)^{-1}Z(x,t)\right)dx \\ =& \frac{1}{\mu_0 a^2}\int_0^1 Y(x,t)f(x)\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)R(x)Y(x,t)\right)dx. \end{aligned}$$

Applying integration by parts twice

$$\dot{V}_3(Z(\cdot,t)) = \int_0^1 \frac{\eta_{\|}(x,t)}{\mu_0 a^2}B_1(x)Y(x,t)^2dx + \frac{\eta_{\|}(1,t)}{\mu_0 a^2}B_2Y(1,t)^2. \tag{9.11}$$

Since $\dot{V}(Z(\cdot,t)) = 2\dot{V}_1(Z(\cdot,t)) + 2\dot{V}_2(Z(\cdot,t)) + 2\dot{V}_3(Z(\cdot,t))$, using Equations (9.9)-(9.11), we obtain

$$\dot{V}(Z(\cdot,t)) = \int_0^1 \frac{\eta_\|(x,t)}{\mu_0 a^2} \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix}^T \Omega(x,\lambda) \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix} dx$$
$$+ \frac{\eta_\|(1,t)}{\mu_0 a^2} \left(2A_4 + A_2(1) + 2B_2\right) Y(1,t)^2 + \frac{\eta_\|(1,t)}{\mu_0 a^2} Z_x(1,t) Y(1,t).$$

Consequently

$$\begin{aligned}
&\dot{V}(Z(\cdot,t)) - \frac{1}{\gamma}\|u(\cdot,t)\|^2_{L_2(0,1)} + \gamma\|Z(\cdot,t)\|^2_{L_2^{M-2}(0,1)} \\
&= \dot{V}(Z(\cdot,t)) - \frac{1}{\gamma}\|u(\cdot,t)\|^2_{L_2(0,1)} + \gamma\|Y(\cdot,t)\|^2_{L_2(0,1)} \\
&= \int_0^1 \frac{\eta_\|(x,t)}{\mu_0 a^2} \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix}^T \Omega(x,\lambda) \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix} dx \\
&\quad + \int_0^1 \frac{\eta_\|(x,t)}{\mu_0 a^2} \left(-\frac{\mu_0 a^2}{\eta_\|(x,t)} \frac{u(x,t)^2}{\gamma} + \frac{\mu_0 a^2}{\eta_\|(x,t)} \gamma Y(x,t)^2 \right) dx \\
&\quad + \frac{\eta_\|(1,t)}{\mu_0 a^2} \left(2A_4 + A_2(1) + 2B_2\right) Y(1,t)^2 + \frac{\eta_\|(1,t)}{\mu_0 a^2} Z_x(1,t) Y(1,t).
\end{aligned} \tag{9.12}$$

Since $\eta_\|(x,t) = a(t)e^{\lambda(t)x}$, $\underline{a} \le \bar{a}e^{\bar{\lambda}}$ for all $(x,t) \in [0,1] \times [0,T]$. Thus

$$\begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix}^T \Omega(x,\lambda) \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix} - \frac{\mu_0 a^2}{\eta_\|(x,t)} \frac{u(x,t)^2}{\gamma} + \frac{\mu_0 a^2}{\eta_\|(x,t)} \gamma Y(x,t)^2$$

$$\leq \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix}^T \Omega(x,\lambda) \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix} - \frac{\mu_0 a^2}{\bar{a}e^{\bar{\lambda}}} \frac{u(x,t)^2}{\gamma} + \frac{\mu_0 a^2}{\underline{a}} \gamma Y(x,t)^2$$

$$= \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix}^T [\Omega(x,\lambda) + \Theta] \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix}.$$

Since $\Omega(x,\lambda) + \Theta \leq 0$, for all $(x,\lambda) \in [0,1] \times [\underline{\lambda}, \bar{\lambda}]$, we conclude that

$$\int_0^1 \frac{\eta_\parallel(x,t)}{\mu_0 a^2} \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix}^T \Omega(x,\lambda) \begin{bmatrix} Y_x(x,t) \\ Y(x,t) \\ u(x,t) \end{bmatrix} dx$$
$$+ \int_0^1 \frac{\eta_\parallel(x,t)}{\mu_0 a^2} \left(-\frac{\mu_0 a^2}{\eta_\parallel(x,t)} \frac{u(x,t)^2}{\gamma} + \frac{\mu_0 a^2}{\eta_\parallel(x,t)} \gamma Y(x,t)^2 \right) dx \leq 0, \tag{9.13}$$

for all $t \geq 0$. Similarly, since from the theorem statement we have $2A_4 + A_2(1) + 2B_2 \leq 0$ and hence

$$\frac{\eta_\parallel(1,t)}{\mu_0 a^2} \left(2A_4 + A_2(1) + 2B_2\right) Y(1,t)^2 \leq 0. \tag{9.14}$$

Using Equation (9.5) and the fact that $Y(x,t) = M(x)^{-1} Z(x,t)$, we get that

$$\frac{\eta_\parallel(1,t)}{\mu_0 a^2} Z_x(1,t) Y(1,t) = -\frac{\eta_\parallel(1,t)}{\mu_0 a^2} Z(1,t) Y(1,t) = -\frac{\eta_\parallel(1,t)}{\mu_0 a^2} M(1) Y(1,t)^2. \tag{9.15}$$

Combining Equations (9.12)-(9.15) we get

$$\dot{V}(Z(\cdot,t)) \leq \frac{1}{\gamma} \|u(\cdot,t)\|^2_{L_2(0,1)} - \gamma \|Z(\cdot,t)\|^2_{L_2^{M^{-2}}(0,1)}.$$

Lemma 9.1 then completes the proof. $\square$

By using sum-of-squares to maximize γ in the conditions of Theorem 9.2, we can minimize the upper bound on the state Z. Because bootstrap current density is inversely proportional to Z and is non-zero on non-zero measure subsets on $[0, 1]$, for all $t \geq 0$, this implies that our controller will maximize the norm of the bootstrap current density.

9.3.1 Constraints on the Control Input. The controller given by Theorem 9.2 will have a spatial distribution which is a function of the state $Z(x, t)$. Unfortunately, this distribution may not correspond to the Gaussian distribution described in our discussion of Subsection 9.1.1. In order to constrain the input profile to have the required Gaussian shape, we propose the following simple heuristic.

To ensure that j_{lh} resembles a Gaussian defined by suitable choice pf the time-varying parameters v_{lh}, μ_{lh} and σ_{lh}, we add an additional constraint to our optimization problem. This constraint has the form

$$g_1(x) \leq j_{lh}(x, t) = \frac{K(x)}{R_0 \mu_0 a^2} Z(x, t) \leq g_2(x),$$

where $g_1(x) < g_2(x)$, for all $x \in [0, 1]$, are polynomial approximations of two selected feasible Gaussians. Since both $K(x)$ and $Z(x, t)$ are continuous, the control is a continuous function bounded by the shape of the constraint envelope defined by $g_1(x)$ and $g_2(x)$. Additionally, we assume that

$$Z(x, t) = \alpha(t) Z_1(x) + (1 - \alpha(t)) Z_2(x), \text{ for all } t \geq 0,$$

where $\alpha \in [0, 1]$ and $Z_1(x)$ is the polynomial approximation of the open loop steady state. Similarly, $Z_2(x)$ is the polynomial approximation of the closed loop steady state under maximum actuation of j_{lh}. Hence, $Z_1(x)$ and $Z_2(x)$ define the expected envelope on the state $Z(x, t)$ established for a given set of operating conditions. The parameter α reflects the actuation capabilities. Since $K(x) = R(x)/M(x)$, the shape constraint becomes

$$R_0\mu_o a^2 M(x) g_1(x) \leq R(x)\left(\alpha Z_1(x) + (1-\alpha) Z_2(x)\right) \leq R_0\mu_o a^2 M(x) g_2(x),$$

for all $(x,\alpha) \in [0,1]\times[0,1]$. Although this approach is only a heuristic, we may improve our results by trying different constraint envelopes, as represented by $g_1(x)$ and $g_2(x)$.

9.3.2 Computation. Finally, we implement the conditions of Theorem 9.2 and the heuristic discussed previously using sum-of-squares polynomials. We formulate the optimization problem as follows. We are given polynomials $Z_1(x)$, $Z_2(x)$, $g_1(x)$ and $g_2(x)$ and solve the following.

Maximize $\gamma > 0$ such that there exist polynomials $M(x)$ and $R(x)$ satisfying

$$\begin{aligned}
&M(x) > 0, \text{ for all } x \in [0,1],\\
&\Omega(x,\lambda) + \Theta \leq 0, \text{ for all } (x,\lambda) \in [0,1]\times[\underline{\lambda},\bar{\lambda}],\\
&2A_4 + 2B_2 + A_2(1) \leq 0, \text{ and}\\
&R_0\mu_o a^2 M(x) g_1(x) \leq R(x)\left(\alpha Z_1(x) + (1-\alpha) Z_2(x)\right) \leq R_0\mu_o a^2 M(x) g_2(x),\\
&\hspace{18em}\text{for all } (x,\alpha) \in [0,1]\times[0,1],
\end{aligned}$$

where $\Omega(x,\lambda)$, Θ, A_4, $A_2(x)$ and B_2 are defined in Theorem 9.2.

We solve the optimization problem using SOSTOOLS. The search for the maximum γ is performed using the bisection method. We solve this problem for the *Tore Supra* tokamak for which $R_0 = 2.38m$ and $a = 0.38m$. Moreover, the plasma resistivity is defined as $\eta_{\|}(x,t) = a(t)e^{\lambda(t)x}$, where $a(t) \in [0.0093, 0.0121]$ and $\lambda(t) \in [4, 7.3]$ for all $t \geq 0$. These values were obtained from the data for shot TS 35109.

9.4 Numerical Simulation

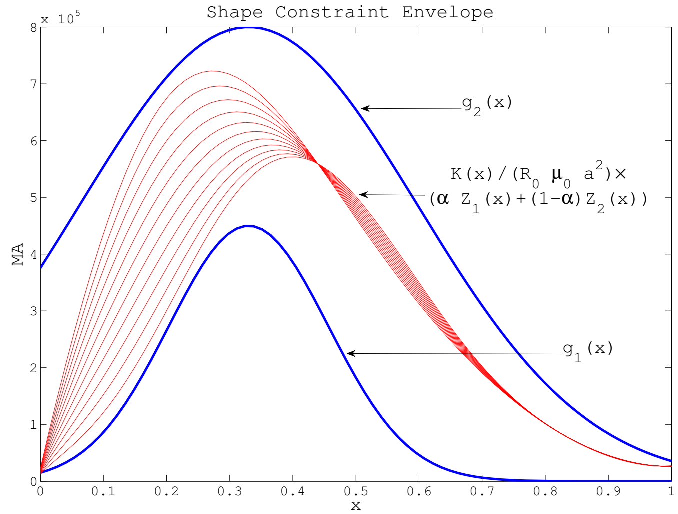


Figure 9.1. Constraint envelope and $\frac{K(x)}{R_0\mu_0 a^2}(\alpha Z_1(x) + (1-\alpha)Z_2(x))$ for $\alpha \in [0,1]$.

We obtain a maximum value of $\gamma = 10^4$ as the solution for the optimization problem for *Tore Supra*. The feasible polynomials $M(x)$ and $R(x)$ obtained for this value of γ are of degree 12 in x. We simulate the closed loop system on the simulator developed in [47]. This simulator considers rge nonlinear evolution model of Z. The following figures provide the simulation results and show that although our controller was developed using a linearized model, it is effective in controlling the nonlinear PDE.

Figure 9.1 shows the constraint envelope as well as $\frac{K(x)}{R_0\mu_0 a^2}(\alpha Z_1(x) + (1-\alpha)Z_2(x))$ for several values of $\alpha \in [0,1]$, where $K(x) = R(x)/M(x)$.

Figure 9.2 shows the comparison between the time evolution of the spatial $L_2(0,1)$ norm of $Z(x,t)$ using both open-loop and closed-loop with closed loop control starting at $t = 12$. Figure 9.3 shows the evolution of the spatial L_2-norm of $j_{bs}(x,t)$ using both open-loop and closed-loop with closed loop control starting at $t = 12$. As a consequence of the decrease in $Z(x,t)$, we are able to obtain a percentage increase of $\approx 90\%$ in $\|j_{bs}(\cdot,t)\|$.

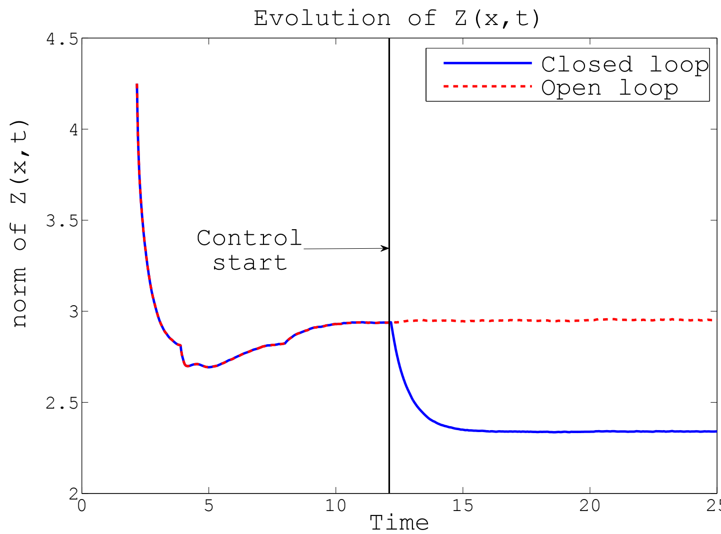


Figure 9.2. Evolution of closed loop ($t \geq 12$) and open loop $\|Z(\cdot, t)\|$.

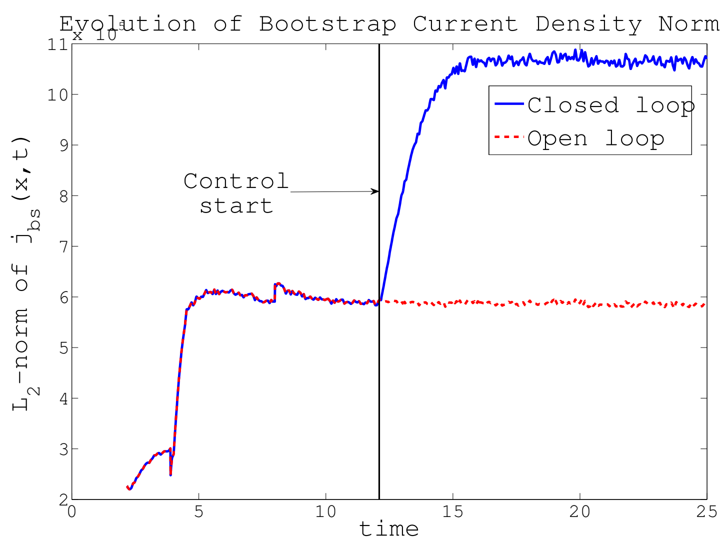


Figure 9.3. Evolution of closed loop ($t \geq 12$) and open loop $\|j_{bs}(\cdot, t)\|$

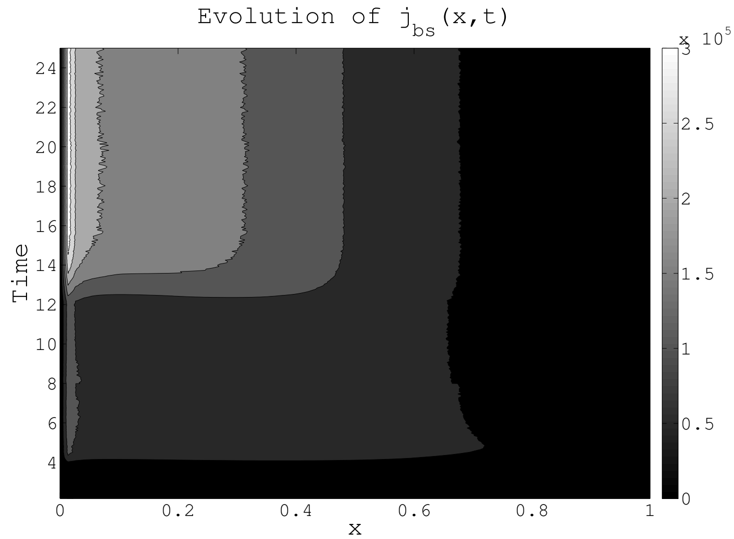


Figure 9.4. Evolution of level sets of bootstrap current density $j_{bs}(x,t)$ in closed loop ($t \geq 12$)

Figure 9.4 illustrates the time evolution of the $j_{bs}(x,t)$ using level sets (shading).

Finally, to analyze the control input shapes, we fit a feasible Gaussian to control input at a time instance as shown in Figure 9.5. We observe that the control input approximates the shape of feasible Gaussians satisfactorily for roughly 70% of the spatial domain. However, the control input departs from the Gaussian shapes as $x \to 0$. This is due to the controller having the form $j_{lh}(x,t) = K(x)Z(x,t)/R_0\mu_0 a^2$ and the boundary condition $Z(0,t) = 0$. Note that the Gaussian approximation of the LHCD current deposit is obtained from hard X-ray measurements and, as stated in [47], a large uncertainty remains concerning the actual deposit close to the plasma center ($x = 0$). If a true zero boundary condition for the input is desired, then RF-antennas (ECCD) can be used to generate a sharper deposit profile near the plasma center.

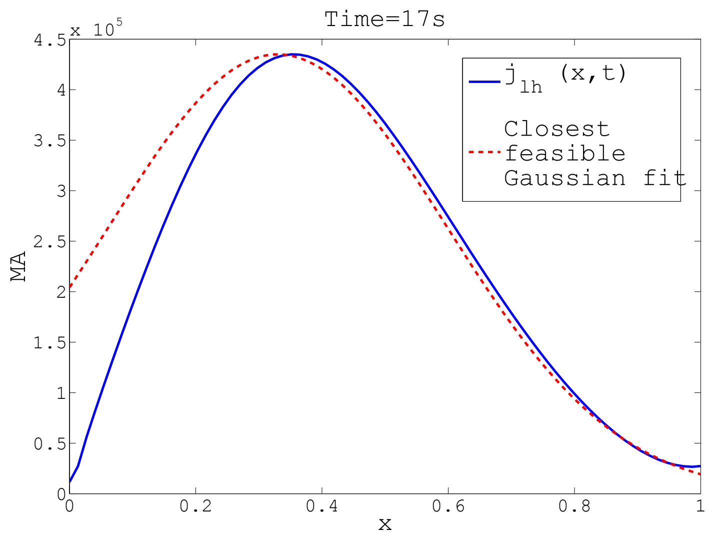


Figure 9.5. Shape comparison between constructed $j_{lh}(x,t)$ and a feasible Gaussian with parameters $v_{lh} = 4.35 \times 10^5$, $\mu_{lh} = 0.33$ and $\sigma_{lh} = 0.072$ at a time instance of $17s$.

CHAPTER 10

CONCLUSION

In this work we considered the analysis and controller and observer synthesis for parabolic PDEs using Sum-of-Squares (SOS) polynomials. In Chapters 5-7 we considered a general class of Parabolic PDEs. Whereas, in Chapters 8-9 we considered the PDE governing the evolution of the poloidal magnetic flux in a Tokamak.

In Chapter 5 we analyze the stability of

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t),$$

with boundary conditions

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{and} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = 0.$$

Here a, b and c are polynomial functions of $x \in [0,1]$. Additionally,

$$|\nu_1| + |\nu_2| > 0 \quad \text{and} |\rho_1| + |\rho_2| > 0. \tag{10.1}$$

Different values of these scalars may be used to represent Dirichlet, Neumann, Robin or mixed boundary conditions.

We establish the exponential stability by constructing Lyapunov functions of the form $V(w(\cdot,t)) = \langle w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle$, where

$$\left(\mathcal{P}y\right)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1), \tag{10.2}$$

and $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ for some $\epsilon > 0$. The results of the numerical experiments presented prove that the presented methodology has an inconsequential amount of conservatism.

In Chapter 6 we construct exponentially stabilizing state feedback based controllers for

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t),$$

with boundary conditions

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{and} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t).$$

Here $u(t) \in \mathbb{R}$ is the control input. Using Lyapunov functions of the form $V(w(\cdot,t)) = \langle w(\cdot,t), \mathcal{P}^{-1} w(\cdot,t)\rangle$, where $\mathcal{P}$ is of the form given in Equation (10.2), we synthesize controllers $\mathcal{F} : H^2(0,1) \to \mathbb{R}$ such that if the control is given by

$$u(t) = \mathcal{F} w(\cdot,t),$$

then the system is exponentially stable. Numerical experiments indicate that the method is very effective in stabilizing systems which are controllable in some appropriate sense. Moreover, we extend the methodology to construct L_2 optimal boundary controllers which minimize the effect of an exogenous distributed input on the state of the system.

In Chapter 7 we construct exponentially estimating state observers for

$$w_t(x,t) = a(x) w_{xx}(x,t) + b(x) w_x(x,t) + c(x) w(x,t),$$

with boundary conditions

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{and} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = u(t).$$

We assume that a boundary measurement (output) of the form

$$y(t) = \mu_1 w(1,t) + \mu_2 w_x(1,t),$$

is available. The goal is to estimate the state w of the system using the boundary output y. For this purpose we design Luenberger observers of the form

$$\hat{w}_t(x,t) = a(x)\hat{w}_{xx}(x,t) + b(x)\hat{w}_x(x,t) + c(x)\hat{w}(x,t) + p(x,t),$$

with boundary conditions

$$\hat{w}_1 w(0,t) + \nu_2 \hat{w}_x(0,t) = 0 \quad \text{and} \quad \rho_1 \hat{w}(1,t) + \rho_2 \hat{w}_x(1,t) = u(t) + q(t).$$

Here $p(x,t)$ and $q(t)$ are the observer inputs.

By constructing Lyapunov functions of the form

$$V((\hat{w}-w)(\cdot,t)) = \langle (\hat{w}-w)(\cdot,t), \mathcal{P}(\hat{w}-w)(\cdot,t)\rangle\,,$$

we construct operator $\mathcal{O}:\mathbb{R}\to L_2(0,1)$ and scalar O such that if

$$p(x,t) = \left(\mathcal{O}(\hat{y}(t)-y(t))\right)(x) \quad \text{and} \quad q(t) = O(\hat{y}(t)-y(t)),$$

where $\hat{y}(t) = \mu_1\hat{w}(1,t) + \mu_2\hat{w}_x(1,t)$, then $\hat{w}\to w$ exponentially fast. Additionally, we show that the observers designed can be coupled to the controllers designed in Chapter 6 to construct exponentially stabilizing observer based boundary controllers. The numerical results indicate that the proposed method is effective in constructing output feedback controllers.

In Chapters 8-9 we consider the gradient of the poloidal magnetic flux $Z=\psi_x$ whose evolution is governed by

$$\frac{\partial Z}{\partial t}(x,t) = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{\|}(x,t)}{x}\frac{\partial}{\partial x}\left(xZ(x,t)\right)\right) + R_0\frac{\partial}{\partial x}\left(\eta_{\|}(x,t)j_{lh}(x,t) + j_{bs}(x,t)\right),$$

with boundary conditions

$$Z(0,t)=0 \text{ and } Z(1,t) = -R_0\mu_0 I_p(t)/2\pi,$$

where

$\eta_{\|}$ = parallel resistivity,

j_{lh} = Lower Hybrid Current Density (LHCD),

j_{bs} = bootstrap current density,

I_p = total plasma current, and

μ_0 = permeability of free space.

In Chapter 8 we regulate the magnetic field line pitch, also known as the safety factor profile, or the q-profile using j_{lh} as the control input. Since

$$q \propto \frac{1}{Z},$$

we regulate the Z-profile. We accomplish this task by using a Lyapunov function of the from

$$V(Z(\cdot,t)) = \int_0^1 x^2(1-x)M(x)^{-1}Z(x,t)^2 dx,$$

where $M(x)$ is a strictly positive polynomial and

$$j_{lh}(x,t) = K_1(x)Z(x,t) + \frac{\partial}{\partial x}\left(K_2(x)Z(x,t)\right),$$

where K_1 and K_2 are rational functions.

In Chapter 9 we maximize the norm of the bootstrap current density j_{bs}. Since

$$j_{bs} \propto \frac{1}{Z},$$

we minimize the norm of the Z-profile. We use a Lypaunov function of the form

$$V(Z(\cdot,t)) = \int_0^1 x^2 M(x)^{-1}Z(x,t)^2 dx,$$

where $M(x)$ is a strictly positive polynomial and

$$j_{lh}(x,t) = K_1(x)Z(x,t),$$

where K_1 is a rational function. Moreover, we present a heuristic such that shape constraints on the control input j_{lh} are respected.

APPENDICES

APPENDIX A

PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

Consider n variables $x_1, \cdots, x_n$, $x_j \in \Omega \subset \mathbb{R}$, $j \in \{1, \cdots, n\}$, and quantity $w(x_1, \cdots, x_n)$, $w : \Omega \times \cdots \times \Omega \to \mathbb{R}$. A general one dimensional Partial Differential Equation (PDE) model is of the form [31]:

$$F\left(x_1, \cdots, x_n, \frac{\partial w}{\partial x_1}, \cdots, \frac{\partial w}{\partial x_n}, \frac{\partial^2 w}{\partial x_1 x_2}, \cdots, \frac{\partial^{(i)} w}{\partial x_1^{(i)}}, \cdots\right) = 0, \tag{A.1}$$

where $F : \Omega \times \cdots \times \Omega \times \mathbb{R} \times \cdots \times \mathbb{R} \to \mathbb{R}$, $\frac{\partial w}{\partial x_j}$, $j \in \{1, \cdots, n\}$, denote the partial derivative of $w(x_1, \cdots, x_n)$ with respect to x_j and $i \in \mathbb{N}$. PDEs are classified in three ways: order, (non)linearity and type. The order of a PDE is defined by the highest order partial derivative appearing in F. For example, Equation (A.1) illustrates an i^{th} order PDE. PDEs can be further classified as linear or nonlinear [32]. To explain this classification, consider a first order PDE in two independent variables x and t and a dependent variable $w(x, t)$ given by

$$F(x, t, w, w_x, w_t) = 0, \tag{A.2}$$

where w_x and w_t denote $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial t}$ respectively. If F is linear, it can be written as

$$F(x, t, w, w_x, w_t) = a(x, t)w_t(x, t) + b(x, t)w_x(x, t) + c(x, t)w(x, t) + d(x, t) = 0. \tag{A.3}$$

Hence, PDE (A.2) is linear if it is linear in the dependent variable and its partial derivatives but not necessarily in the independent variables. If F is not linear in the dependent variable or in its partial derivatives, PDE (A.2) is nonlinear. Nonlinear PDEs can be further classified as semi-linear or quasi-linear [31], [32]. A PDE of the form

$$F(x, t, w, w_x, w_t) = a(x, t)w_t(x, t) + b(x, t)w_x(x, t) + c(x, t, w) = 0, \tag{A.4}$$

where c is non-linear in w, is known as a *semi-linear* PDE. The function F is linear in w_t and w_x but non-linear in w. An equation of the form

$$F(x, t, w, w_x, w_t) = a(x, t, w)w_t(x, t) + b(x, t, w)w_x(x, t) + c(x, t, w) = 0, \tag{A.5}$$

is called *quasi-linear*. Thus, a quasi-linear PDE has coefficients which are functions of both the independent and dependent variables.

To explain the classification of PDEs by type, consider the following general second order PDE in two independent variables

$$F(x,t,w,w_x,w_t,w_{xt},w_{xx},w_{tt}) = aw_{tt} + bw_{xt} + cw_{xx} + dw_t + ew_x + fw + g = 0, \tag{A.6}$$

where the coefficients are functions of the independent variables x and t only. The type of a second order PDE depends on the discriminant defined as

$$\Delta = b^2 - 4ac. \tag{A.7}$$

Under the assumption that the discriminant does not change sign in some region Ω, the PDE (A.6) is one of the following types in Ω:

$$\Delta > 0: \qquad \textit{hyperbolic}, \tag{A.8}$$

$$\Delta = 0: \qquad \textit{parabolic}, \tag{A.9}$$

$$\Delta < 0: \qquad \textit{elliptic}. \tag{A.10}$$

If the discriminant Δ changes sign in the region Ω, the PDE is said to be of a *mixed type* in Ω.

For the Equation (A.6), let us assume that $x \in \Omega \subset \mathbb{R}^n$, Ω open. Additionally, assume that the variable t represents time, thus, $t \geq 0$. Then, the PDE given by Equation (A.6) is often known as an *evolution equation* because the quantity $w(x,t)$ evolves in time from a given initial configuration $w(x,0) = w_0(x)$. The function $w_0(x)$ is known as the *initial condition*. If the quantity w is scalar valued for each x and t, that is, $w : \Omega \times [0,\infty) \to \mathbb{R}$, then the PDE is known as a *scalar valued PDE.*

Let $\partial\Omega$ denote the boundary[6] of Ω. Then, for an operator $\mathcal{G}$, a constraint of

[6] $\partial\Omega = \bar{\Omega}\backslash\Omega$, where $\bar{\Omega}$ is the closure of Ω.

the form

$$\left(\mathcal{G}w\right)(x,t) = f(x,t), \quad \text{for} \quad x \in \partial\Omega, \quad t \in [0,\infty), \tag{A.11}$$

is known as a *boundary condition.* Boundary conditions can be classified based on the operator $\mathcal{G}$. If $\left(\mathcal{G}w\right)(x,t) = w(x,t)$, $x \in \partial\Omega$, then the boundary condition is known as a *Dirichlet boundary condition.* A condition of the form $\left(\mathcal{G}w\right)(x,t) = \nabla_x w(x,t) \cdot \hat{n}$, $x \in \partial\Omega$, where ∇_x denotes the gradient with respect to x and $\hat{n}$ is the unit outward normal vector, is called a *Neumann boundary condition.* Of course, this requires that the boundary be such that the outward normal vector can be specified. A linear combination of Dirichlet and Neumann boundary conditions is known as a *Robin boundary condition.* A PDE can can have different boundary conditions on different sections of the boundary $\partial\Omega$.

A.1 Well-Posedness of Parabolic PDEs

The research work presented in the thesis deals with evolution equations given by scalar valued parabolic PDEs. Parabolic PDEs are used to model processes such as diffusion, transport and reaction. An example of a fairly well known parabolic PDE is the heat equation. For a uniform one dimensional rod of length L, the temperature of the rod $w(x,t)$ at any point $x \in [0,L]$ and at time $t > 0$ is governed by the heat equation given by

$$w_t(x,t) = \kappa w_{xx}(x,t), \tag{A.12}$$

where κ is the thermal conductivity of the material of the rod. It is clear from Equation (A.9) that the PDE (A.12) is of the parabolic type. Further examples of parabolic PDEs are the equations modeling the evolution of the poloidal magnetic flux in a tokamak ψ and its gradient ψ_x given by Equations (4.2) and (4.6) in Chapter 4.

The first question to be asked of a parabolic PDE, or in fact any type of PDE, is if it is *well-posed.* A parabolic PDE is well-posed if:

1. the PDE has a unique solution;

2. the solution depends continuously on the data given in the problem.

A.1.1 Semigroup theory. The definition of a solution of a PDE is non-trivial [31], [32], [33], [34]. One way of establishing the definitions of solutions and their uniqueness and existence is by using semigroup theory.

Consider the following second order inhomogeneous parabolic PDE

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t) + f(x,t) \tag{A.13}$$

with Dirichlet boundary conditions,

$$w(0,t) = 0 \quad \text{and} \quad w(1,t) = 0, \tag{A.14}$$

where the functions a, b and c are C^1 functions, f is a known function, $x \in [0,1]$ and $t \geq 0$. We can write this PDE as a differential equation as follows. Let

$$w(t) = w(\cdot,t) \quad \text{and} \quad f(t) = f(\cdot,t).$$

Additionally, define the following differential operator

$$\mathcal{A} = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} + c(x). \tag{A.15}$$

Then, the PDE (A.13) can be written as

$$\dot{w}(t) = (\mathcal{A}w)(t) + f(t). \tag{A.16}$$

Let us denote by $\mathcal{D}_{\mathcal{A}}$ the space of functions over which the operator $\mathcal{A}$ is well defined and also incorporates the boundary conditions (A.14). Thus

$$\mathcal{D}_{\mathcal{A}} = \{w \in H^2(0,1) : w(0) = 0 \quad \text{and} \quad w(1) = 0\}. \tag{A.17}$$

Under certain conditions, the pair $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ is associated with an operator valued function $S(t)$ called the *strongly continuous semigroup generated by* $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$.

Definition A.1. *A* ***strongly continuous semigroup****, or a* $\boldsymbol{C_0}$***-semigroup*** *is an operator valued function* $S(t)$*,* $S : [0, \infty) \to \mathcal{L}(L_2(0,1))$*, that satisfies*

$$
\begin{aligned}
&S(t+s) = S(t)S(s), \quad \textit{for } t, s \geq 0;\\
&S(0) = \mathcal{I};\\
&\|S(t)y - y\| \to 0 \quad \textit{as } t \to 0^+ \textit{ for all } y \in L_2(0,1).
\end{aligned}
$$

Of course, the question arises whether the pair $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ generates a C_0-semigroup. This question can be answered using the Hille-Yoisida Theorem [45, Theorem 2.1.12].

Using the semigroup theory, we can discuss the uniqueness and existence of solutions. We begin with the following notion of a solution.

Definition A.2. *A function* $w(t)$ *is a* ***classical solution of*** (A.16) ***on*** $\boldsymbol{[0, \tau]}$ *if* $z \in C^1([0,\tau]; L_2(0,1))$*,* $z(t) \in \mathcal{D}_{\mathcal{A}}$ *for all* $t \in [0, \tau]$ *and* $z(t)$ *satisfies* (A.16) *for all* $t \in [0, \tau]$*.*

The function $z(t)$ *is a* ***classical solution of*** (A.16) ***on*** $\boldsymbol{[0, \infty]}$ *if it is a classical solution on* $[0, \tau]$ *for every* $\tau \geq 0$*.*

A classical solution captures all the properties that one might expect a 'solution' of the PDE (A.13) to possess. That is, the solution is continuously differentiable in time, its spatial derivatives up to order 2 are continuous, satisfies the equation and the boundary conditions. The following theorem establishes the existence of a unique classical solution of PDE (A.13) using the semigroup theory.

Theorem A.3. *[45, Theorem 3.1.3] If the operator* $\mathcal{A}$ *generates a* C_0*-semigroup* $S(t)$ *on* $L_2(0,1)$*,* $f \in C^1([0,\tau]; L_2(0,1))$ *and* $w(0) = w_0 \in \mathcal{D}_{\mathcal{A}}$*. Then there exists a unique classical solution of PDE* (A.13) *given by*

$$
w(t) = S(t)w_0 + \int_0^t S(t-s)f(s)ds. \tag{A.18}
$$

The condition that $f \in C^1([0,\tau]; L_2(0,1))$ is very conservative. In fact, it can be weakened to $f \in L_2([0,\tau]; L_2(0,1))$ with $w_0 \in L_2(0,1)$, in which case, $w(t)$ in Equation (A.18) is known as the *mild solution* or the *weak solution*.

Corollary A.4. *If the operator $\mathcal{A}$ generates a C_0-semigroup $S(t)$ on $L_2(0,1)$, $f \in L_2([0,\tau]; L_2(0,1))$ and $w(0) = w_0 \in L_2(0,1)$. Then there exists a unique weak solution of PDE* (A.13) *given by*

$$w(t) = S(t)w_0 + \int_0^t S(t-s)f(s)ds. \tag{A.19}$$

Simply put, the idea is that the weak solution satisfies the PDE (A.13) almost everywhere in t and x, that is, under the integral. Thus, instead of searching for solutions which are continuously differentiable in x and t, we can search over the larger space of functions whose *generalized derivatives* or *weak derivatives* exist. Refer to Chapters 5 and 7 in [31] for weak derivatives and weak solutions of parabolic PDEs.

For the homogeneous case ($f = 0$), the classical solution of PDE (A.13) is given by

$$w(t) = S(t)w_0, \quad w_0 \in \mathcal{D}_{\mathcal{A}}.$$

Compare this to the solution of the ODE $\dot{x}(t) = Ax(t)$, $A \in \mathbb{R}^{n\times n}$, which is given by

$$x(t) = e^{At}x_0, \quad x_0 \in \mathbb{R}^n.$$

This comparison immediately illustrates that a C_0-semigroup can be thought of as an infinite dimensional generalization of the matrix exponential.

Note that although we chose Dirichlet boundary conditions in Equation (A.14) to illustrate the uniqueness and existence of solutions, the same theory applies to Neumann and Robin boundary conditions.

Remark A.5. *Establishing the well-posedness of parabolic PDEs using semigroup theory requires that the coefficients a, b, c in Equation* (A.13) *be independent of t. If* *this is not the case, the Galerkin method [31, Section 7.1] may be used to establish the existence and uniqueness of weak solutions.*

A.2 Stability of systems governed by Parabolic PDEs

Once we have established that the PDE (A.13) has a classical (weak) solution, we would like to know if the PDE is stable. We begin by defining the following notion of stability.

Definition A.6. *Suppose that $w(t)$ is a classical (weak) solution of* (A.13) *with initial condition w_0. Then, the PDE is **exponentially stable** if for any w_0, there exist scalars $M, \omega > 0$ such that*

$$\|w(t)\| \leq Me^{-\omega t}, \quad t \geq 0. \tag{A.20}$$

Exponential stability can be established using semigroup theory.

Definition A.7. *A C_0-semigroup $S(t)$ on $L_2(0,1)$ is **exponentially stable** if there exist scalars $N, \alpha > 0$ such that*

$$\|S(t)\|_{\mathcal{L}(L_2(0,1))} \leq Ne^{-\alpha t}, \quad t \geq 0. \tag{A.21}$$

The following theorem may be used to verify the exponential stability of C_0-semigroups.

Theorem A.8. *[45, Theorem 5.1.3] Suppose that the pair $(\mathcal{A}, \mathcal{D}_\mathcal{A})$ generates a C_0-semigroup $S(t)$ on $L_2(0,1)$. Then $S(t)$ is exponentially stable if and only if there exists $\mathcal{P} \in \mathcal{L}(L_2(0,1))$ such that*

$$\langle y, \mathcal{P}y \rangle > 0, \quad \textit{for all } y \in \mathcal{D}_\mathcal{A}, \quad y \neq 0 \tag{A.22}$$

$$\langle \mathcal{A}y, \mathcal{P}y \rangle + \langle \mathcal{P}\mathcal{A}y, y \rangle = -\|y\|^2, \quad \textit{for all } y \in \mathcal{D}_\mathcal{A}. \tag{A.23}$$

Note that for the PDE (A.13) with $f = 0$, the PDE is exponentially stable if the C_0-semigroup $S(t)$ generated by $(\mathcal{A}, \mathcal{D}_{\mathcal{A}})$ is exponentially stable because

$$\|w(t)\| = \|S(t)w_0\| \leq \|S(t)\|_{\mathcal{L}(L_2(0,1))}\|w_0\| \leq Ne^{-\alpha t}\|w_0\|.$$

Then, by setting $\omega = \alpha$ and $M = N\|w_0\|$ and using Definition A.6 shows that the PDE is exponentially stable.

Exponential stability can also be established by using *Lyapunov functions*. Suppose there exists a classical (weak) solution of PDE (A.13) and a Lyapunov function $V(w(t))$ such that for some $\epsilon, \alpha > 0$

$$V(w(t)) \geq \epsilon\|w(t)\|^2 \tag{A.24}$$

$$\dot{V}(w(t)) \leq -\alpha V(w(t)). \tag{A.25}$$

Then, by integrating the second inequality in time and using the first inequality, we can show that the PDE is exponentially stable. Note that if we choose $V(w(t)) = \langle w(t), \mathcal{P}w(t)\rangle$, for some positive operator $\mathcal{P} \in \mathcal{L}\left(L_2(0,1)\right)$, it becomes evident that Inequalities (A.24)-(A.25) are similar to Inequalities (A.22)-(A.23).

APPENDIX B

UPPER BOUNDS FOR OPERATOR INEQUALITIES

First, recall the variation of Wirtinger's inequality

Lemma B.1. *[92, 36] For any $w \in H^1(0,1)$*

$$\int_0^1 w(x)^2 dx \leq w(0)^2 + \frac{4}{\pi^2}\int_0^1 w_x(x)^2 dx.$$

Now recall the definition of $\mathcal{M}$ from Chapter 5.

Definition B.2. *We say*

$$\{Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}\} = \mathcal{M}(M, K_1, K_2)$$

if the following hold

$$\begin{aligned}
Q_0(x) =& \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(a(x)M(x)) - b(x)M(x)\right) + 2M(x)c(x) - \frac{\alpha\epsilon\pi^2}{2} \\
&+ 2\left[\frac{\partial}{\partial x}\left[a(x)\left(K_1(x,\xi) - K_2(x,\xi)\right)\right]\right]_{\xi=x}, \\
Q_1(x,\xi) =& \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[a(x)K_1(x,\xi)\right] - b(x)K_1(x,\xi)\right) + c(x)K_1(x,\xi) \\
&+ \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left[a(\xi)K_1(x,\xi)\right] - b(\xi)K_1(x,\xi)\right) + c(\xi)K_1(x,\xi), \\
Q_2(x,\xi) =& \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[a(x)K_2(x,\xi)\right] - b(x)K_2(x,\xi)\right) + c(x)K_2(x,\xi) \\
&+ \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left[a(\xi)K_2(x,\xi)\right] - b(\xi)K_2(x,\xi)\right) + c(\xi)K_2(x,\xi), \\
Q_3(x) =& 2n_5 a(1)K_1(1,x), \\
Q_4(x) =& -2n_2 a(0)K_2(0,x), \\
Q_5 =& 2n_6 n_4 a(1)M(1) - n_6^2\left[a_x(1)M(1) + a(1)M_x(1) - b(1)M(1)\right], \\
Q_6 =& 2n_6 n_5 a(1)M(1), \\
Q_7(x) =& K_1(1,x)\left[2n_4 a(1) + 2n_6 b(1)\right] - 2n_6\left[a_x(1)K_1(1,x) + a(1)K_{1,x}(1,x)\right], \\
Q_8 =& -2n_3 n_1 a(0)M(0) \\
&+ n_3^2\left[a_x(0)M(0) + a(0)M_x(0) - b(0)M(0) + \frac{\alpha\epsilon\pi^2}{2}\right], \\
Q_9 =& -2n_3 n_2 a(0)M(0),
\end{aligned}$$

$$Q_{10}(x) = -\,K_2(0,x)\left[2n_1 a(0) + 2n_3 b(0)\right] + 2n_3\left[a_x(0)K_2(0,x) + a(0)K_{2,x}(0,x)\right],$$

where $K_{1,x}(1,x) = \left[K_{1,x}(x,\xi)|_{x=1}\right]_{\xi=x}$, $K_{2,x}(0,x) = \left[K_{2,x}(x,\xi)|_{x=0}\right]_{\xi=x}$ *and* $\epsilon > 0$ *and* n_i, $i \in \{1,\cdots,6\}$, *are scalars.*

Lemma B.3. *Suppose we are given* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$,

$$\{Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9, Q_{10}\} = \mathcal{M}(M, K_1, K_2),$$

and scalars n_i, $i \in \{1,\cdots,6\}$, *as defined in Definition 5.1. Then, for any solution* $w(x,t)$ *of Equations* (5.1)-(5.2), $\mathcal{A}$ *as defined in Equation* (5.6) *and* $\mathcal{P}$ *defined in Equation* (5.12), *we have that*

$$\begin{aligned}
&\langle \mathcal{A}w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle + \langle w(\cdot,t), \mathcal{P}\mathcal{A}w(\cdot,t)\rangle \\
&\le \langle w(\cdot,t), \mathcal{Q}w(\cdot,t)\rangle + w_x(1,t)\int_0^1 Q_3(x)w(x,t)dx + w_x(0,t)\int_0^1 Q_4(x)w(x,t)dx \\
&\quad + w(1,t)\left(Q_5 w(1,t) + Q_6 w_x(1,t) + \int_0^1 Q_7(x)w(x,t)dx\right) \\
&\quad + w(0,t)\left(Q_8 w(0,t) + Q_9 w_x(0,t) + \int_0^1 Q_{10}(x)w(x,t)dx\right),
\end{aligned}$$

where $\mathcal{Q}$ *is defined as*

$$\left(\mathcal{Q}y\right)(x) = Q_0(x)y(x) + \int_0^x Q_1(x,\xi)y(\xi)d\xi + \int_x^1 Q_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1).$$

Proof. We begin by considering the following decomposition

$$\begin{aligned}
&\langle \mathcal{A}w(\cdot,t), \mathcal{P}w(\cdot,t)\rangle + \langle w(\cdot,t), \mathcal{P}\mathcal{A}w(\cdot,t)\rangle \\
&= 2\int_0^1 \left(a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t)\right)\left(\mathcal{P}w\right)(x,t)dx \\
&= 2\left(\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5\right), \qquad \text{(B.1)}
\end{aligned}$$

where

$$\Gamma_1 = \int_0^1 w_{xx}(x,t)a(x)M(x)w(x,t)dx,$$

$$\Gamma_2 = \int_0^1 w_x(x,t)b(x)M(x)w(x,t)dx,$$
$$\Gamma_3 = \int_0^1 w_{xx}(x,t)a(x)\left(\int_0^x K_1(x,\xi)w(\xi,t)d\xi + \int_x^1 K_2(x,\xi)w(\xi,t)d\xi\right)dx,$$
$$\Gamma_4 = \int_0^1 w_x(x,t)b(x)\left(\int_0^x K_1(x,\xi)w(\xi,t)d\xi + \int_x^1 K_2(x,\xi)w(\xi,t)d\xi\right)dx,$$
$$\begin{aligned}\Gamma_5 = &\int_0^1 w(x,t)^2 M(x)c(x)dx + \int_0^1\int_0^x w(x,t)c(x)K_1(x,\xi)w(\xi,t)d\xi \\ &+ \int_0^1\int_x^1 w(x,t)c(x)K_2(x,\xi)w(\xi,t)d\xi.\end{aligned}$$

Applying integration by parts twice

$$\begin{aligned}\Gamma_1 = &-\int_0^1 w_x^2(x,t)a(x)M(x)dx + \int_0^1 w^2(x,t)\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(a(x)M(x)\right)dx, \\ &+ w(1,t)\left(a(1)M(1)w_x(1,t) - \left(\frac{1}{2}a_x(1)M(1) + \frac{1}{2}a(1)M_x(1)\right)w(1,t)\right) \\ &+ w(0,t)\left(-a(0)M(0)w_x(0,t) + \left(\frac{1}{2}a_x(0)M(0) + \frac{1}{2}a(0)M_x(0)\right)w(0,t)\right).\end{aligned} \tag{B.2}$$

Since $a(x)M(x) \geq \alpha\epsilon$, applying a variation of Wirtinger's inequality given in Lemma B.1 produces

$$\begin{aligned}&-\int_0^1 w_x(x,t)^2 a(x)M(x)dx \\ &\qquad \leq -\alpha\epsilon\int_0^1 w_x(x,t)^2 dx \\ &\qquad \leq -\frac{\alpha\epsilon\pi^2}{4}\int_0^1 w(x,t)^2 dx + \frac{\alpha\epsilon\pi^2}{4}\int_0^1 w(0,t)^2 dx.\end{aligned}$$

Substituting into Equation (B.2),

$$\begin{aligned}\Gamma_1 \leq &\int_0^1 w^2(x,t)\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(a(x)M(x)\right) - \frac{\alpha\epsilon\pi^2}{4}\right)dx \\ &+ w(1,t)\left(a(1)M(1)w_x(1,t) - \left(\frac{1}{2}a_x(1)M(1) + \frac{1}{2}a(1)M_x(1)\right)w(1,t)\right) \\ &+ w(0,t)\left(-a(0)M(0)w_x(0,t) + \left(\frac{1}{2}a_x(0)M(0) + \frac{1}{2}a(0)M_x(0) + \frac{\alpha\epsilon\pi^2}{4}\right)w(0,t)\right).\end{aligned}$$

Using the representation of $w(0,t)$, $w_x(0,t)$, $w(1,t)$ and $w_x(1,t)$ given in Definition 5.1, we obtain

$$\begin{aligned}\Gamma_1 \leq &\int_0^1 w^2(x,t)\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(a(x)M(x)\right) - \frac{\alpha\epsilon\pi^2}{4}\right)dx \\ &+ \left(n_6 n_4 a(1)M(1) - \frac{n_6^2}{2}a_x(1)M(1) - \frac{n_6^2}{2}a(1)M_x(1)\right)w(1,t)^2 \\ &+ \left(n_6 n_5 a(1)M(1)\right)w(1,t)w_x(1,t) + \left(-n_3 n_2 a(0)M(0)\right)w(0,t)w_x(0,t) \\ &+ \left(-n_3 n_1 a(0)M(0) + \frac{n_3^2}{2}a_x(0)M(0) + \frac{n_3^2}{2}a(0)M_x(0) + \frac{n_3^2\alpha\epsilon\pi^2}{4}\right)w(0,t)^2.\end{aligned} \tag{B.3}$$

Applying integration by parts once

$$\Gamma_2 = -\int_0^1 w^2(x,t)\frac{1}{2}\frac{\partial}{\partial x}\left(b(x)M(x)\right)dx + w^2(1,t)\frac{n_6^2}{2}b(1)M(1) - w^2(0,t)\frac{n_3^2}{2}b(0)M(0). \tag{B.4}$$

Applying integration by parts twice and using the fact that $K_1(x,x) = K_2(x,x)$,

$$\begin{aligned}\Gamma_3 = &\int_0^1 w^2(x,t)\left[\frac{\partial}{\partial x}\left[a(x)\left(K_1(x,\xi) - K_2(x,\xi)\right)\right]\right]_{\xi=x} dx \\ &+ \int_0^1\int_0^x w(x,t)\frac{\partial^2}{\partial x^2}\left(a(x)K_1(x,\xi)\right)w(\xi,t)d\xi dx \\ &+ \int_0^1\int_x^1 w(x,t)\frac{\partial^2}{\partial x^2}\left(a(x)K_2(x,\xi)\right)w(\xi,t)d\xi dx \\ &+ w_x(1,t)\int_0^1 n_5 a(1)K_1(1,x)w(x,t)dx - w_x(0,t)\int_0^1 n_2 a(0)K_2(0,x)w(x,t)dx \\ &+ w(1,t)\int_0^1\left[n_4 a(1)K_1(1,x) - n_6 a_x(1)K_1(1,x) - n_6 a(1)K_{1,x}(1,x)\right]w(x,t)dx \\ &+ w(0,t)\int_0^1\left[-n_1 a(0)K_2(0,x) + n_3 a_x(0)K_2(0,x) + n_3 a(0)K_{2,x}(0,x)\right]w(x,t)dx.\end{aligned}$$

Applying a change of order of integration in the double integrals, switching between x and ξ and using the fact that $K_1(x,\xi) = K_2(\xi,x)$ produces

$$\Gamma_3 = \int_0^1 w^2(x,t)\left[\frac{\partial}{\partial x}\left[a(x)\left(K_1(x,\xi) - K_2(x,\xi)\right)\right]\right]_{\xi=x} dx$$

$$
\begin{aligned}
&+ \frac{1}{2}\int_0^1\int_0^x w(x,t)\left(\frac{\partial^2}{\partial x^2}\left(a(x)K_1(x,\xi)\right)+\frac{\partial^2}{\partial \xi^2}\left(a(\xi)K_1(x,\xi)\right)\right)w(\xi,t)d\xi dx\\
&+ \frac{1}{2}\int_0^1\int_x^1 w(x,t)\left(\frac{\partial^2}{\partial x^2}\left(a(x)K_2(x,\xi)\right)+\frac{\partial^2}{\partial \xi^2}\left(a(\xi)K_2(x,\xi)\right)\right)w(\xi,t)d\xi dx\\
&+ w_x(1,t)\int_0^1 n_5 a(1)K_1(1,x)w(x,t)dx - w_x(0,t)\int_0^1 n_2 a(0)K_2(0,x)w(x,t)dx\\
&+ w(1,t)\int_0^1 \left[n_4 a(1)K_1(1,x) - n_6 a_x(1)K_1(1,x) - n_6 a(1)K_{1,x}(1,x)\right]w(x,t)dx\\
&+ w(0,t)\int_0^1 \left[-n_1 a(0)K_2(0,x) + n_3 a_x(0)K_2(0,x) + n_3 a(0)K_{2,x}(0,x)\right]w(x,t)dx.
\end{aligned}
\tag{B.5}
$$

Similarly,

$$
\begin{aligned}
\Gamma_4 = &-\int_0^1\int_0^x w(x,t)\left(\frac{1}{2}\frac{\partial}{\partial x}\left(b(x)K_1(x,\xi)\right)+\frac{1}{2}\frac{\partial}{\partial \xi}\left(b(\xi)K_1(x,\xi)\right)\right)w(\xi,t)d\xi dx\\
&-\int_0^1\int_x^1 w(x,t)\left(\frac{1}{2}\frac{\partial}{\partial x}\left(b(x)K_2(x,\xi)\right)+\frac{1}{2}\frac{\partial}{\partial \xi}\left(b(\xi)K_2(x,\xi)\right)\right)w(\xi,t)d\xi dx\\
&+ w(1,t)\int_0^1 n_6 b(1)K_1(1,x)w(x)dx - w(0)\int_0^1 n_3 b(0)K_2(0,x)w(x,t)dx.
\end{aligned}
\tag{B.6}
$$

Finally, changing the order of integration produces

$$
\begin{aligned}
\Gamma_5 = &\int_0^1 w(x,t)^2 M(x)c(x)dx + \int_0^1\int_0^x w(x,t)\left(\frac{1}{2}\left[c(x)+c(\xi)\right]K_1(x,\xi)\right)w(\xi,t)d\xi\\
&+\int_0^1\int_x^1 w(x,t)\left(\frac{1}{2}\left[c(x)+c(\xi)\right]K_2(x,\xi)\right)w(\xi,t)d\xi.
\end{aligned}
\tag{B.7}
$$

Substituting Equations (B.3)-(B.7) into (B.1) produces

$$
\begin{aligned}
&\langle \mathcal{A}w(\cdot,t),\mathcal{P}w(\cdot,t)\rangle + \langle w(\cdot,t),\mathcal{P}\mathcal{A}w(\cdot,t)\rangle\\
&\le \langle w(\cdot,t),\mathcal{Q}w(\cdot,t)\rangle + w_x(1,t)\int_0^1 Q_3(x)w(x,t)dx + w_x(0,t)\int_0^1 Q_4(x)w(x,t)dx\\
&+ w(1,t)\left(Q_5 w(1,t) + Q_6 w_x(1,t) + \int_0^1 Q_7(x)w(x,t)dx\right)\\
&+ w(0,t)\left(Q_8 w(0,t) + Q_9 w_x(0,t) + \int_0^1 Q_{10}(x)w(x,t)dx\right).
\end{aligned}
$$

□

For the following corollary, recall the definition of $\mathcal{J}$ from Chapter 7.

Definition B.4. *We say*

$$\{R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}\} = \mathcal{J}(M, K_1, K_2)$$

if the following hold

$$\begin{aligned}
R_0(x) =& \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(a(x)M(x)) - b(x)M(x)\right) + 2M(x)c(x) - \frac{\alpha\epsilon\pi^2}{2} \\
&+ 2\left[\frac{\partial}{\partial x}\left[a(x)\left(K_1(x,\xi) - K_2(x,\xi)\right)\right]\right]_{\xi=x}, \\
R_1(x,\xi) =& \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[a(x)K_1(x,\xi)\right] - b(x)K_1(x,\xi)\right) + c(x)K_1(x,\xi) \\
&+ \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left[a(\xi)K_1(x,\xi)\right] - b(\xi)K_1(x,\xi)\right) + c(\xi)K_1(x,\xi), \\
R_2(x,\xi) =& \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[a(x)K_2(x,\xi)\right] - b(x)K_2(x,\xi)\right) + c(x)K_2(x,\xi) \\
&+ \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \xi}\left[a(\xi)K_2(x,\xi)\right] - b(\xi)K_2(x,\xi)\right) + c(\xi)K_2(x,\xi), \\
R_3(x) =& -2l_2 a(0)K_2(0,x), \\
R_4 =& -2l_3 l_1 a(0)M(0) \\
&+ l_3^2\left[a_x(0)M(0) + a(0)M_x(0) - b(0)M(0) + \frac{\alpha\epsilon\pi^2}{2}\right], \\
R_5 =& -2l_3 n_2 a(0)M(0), \\
R_6(x) =& -K_2(0,x)\left[2l_1 a(0) + 2l_3 b(0)\right] + 2l_3\left[a_x(0)K_2(0,x) + a(0)K_{2,x}(0,x)\right], \\
R_7 =& -a_x(1)M(1) - a(1)M_x(1) + b(1)M(1), \\
R_8 =& 2a(1)M(1), \\
R_9(x) =& -2a_x(1)K_1(1,x) - 2a(1)K_{1,x}(1,x) + 2b(1)K_1(1,x), \\
R_{10}(x) =& 2a(1)K_1(1,x),
\end{aligned}$$

where $K_{1,x}(1,x) = \left[K_{1,x}(x,\xi)|_{x=1}\right]_{\xi=x}$, $K_{2,x}(0,x) = \left[K_{2,x}(x,\xi)|_{x=0}\right]_{\xi=x}$ *and* $\epsilon > 0$ *and* l_i, $i \in \{1, \cdots, 3\}$, *are scalars.*

Corollary B.5. *Suppose we are given* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$,

$$\{R_0, R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}\} = \mathcal{J}(M, K_1, K_2),$$

and scalars l_i, $i \in \{1, \cdots, 3\}$, *as defined in Definition 7.2. Then, for any solution* $e(x,t)$ *of Equations* (7.15)-(7.16), $\mathcal{A}$ *as defined in Equation* (7.12) *and* $\mathcal{P}$ *defined in Equation* (5.12), *we have that*

$$\begin{aligned}
&\langle \mathcal{A}e(\cdot,t), \mathcal{P}e(\cdot,t)\rangle + \langle e(\cdot,t), \mathcal{P}\mathcal{A}e(\cdot,t)\rangle \\
&\leq \langle e(\cdot,t), \mathcal{R}e(\cdot,t)\rangle + e_x(0,t)\int_0^1 R_3(x)e(x,t)dx \\
&\quad + e(0,t)\left(R_4 e(0,t) + R_5 e_x(0,t) + \int_0^1 R_6(x)e(x,t)dx\right) \\
&\quad + e(1,t)\left(R_7 e(1,t) + R_8 e_x(1,t) + \int_0^1 R_9(x)e(x,t)dx\right) \\
&\quad + e_x(1,t)\int_0^1 R_{10}(x)e(x,t)dx,
\end{aligned}$$

where $\mathcal{R}$ *is defined as*

$$\left(\mathcal{R}y\right)(x) = R_0(x)y(x) + \int_0^x R_1(x,\xi)y(\xi)d\xi + \int_x^1 R_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1).$$

The proof of Corollary B.5 can be established by using Definition 7.2 instead of Definition 5.1 in the proof of Lemma B.3.

Now recall the definition of $\mathcal{N}$ from Chapter 6.

Definition B.6. *We say*

$$\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \mathcal{N}(M, K_1, K_2)$$

if the following hold

$$\begin{aligned}
T_0(x) =& a_{xx}(x)M(x) + a(x)M_{xx}(x) - b_x(x)M(x) + b(x)M_x(x) + 2c(x)M(x) \\
&+ 2a(x)\left[K_{1,x}(x,x) - K_{2,x}(x,x)\right] - \frac{\pi^2\alpha\epsilon}{2},
\end{aligned}$$

$$
\begin{aligned}
T_1(x,\xi) =& [a(x)K_{1,xx}(x,\xi) + a(\xi)K_{1,\xi\xi}(x,\xi)] + [b(x)K_{1,x}(x,\xi) + b(\xi)K_{1,\xi}(x,\xi)] \\
&+ [c(x)K_1(x,\xi) + c(\xi)K_1(x,\xi)]\,, \\
T_2(x,\xi) =& [a(x)K_{2,xx}(x,\xi) + a(\xi)K_{2,\xi\xi}(x,\xi)] + [b(x)K_{2,x}(x,\xi) + b(\xi)K_{2,\xi}(x,\xi)] \\
&+ [c(x)K_2(x,\xi) + c(\xi)K_2(x,\xi)]\,, \\
T_3 =& - m_3\left(a(0)M_x(0) - \frac{\alpha\epsilon\pi^2}{2}\right) + m_3\left(a_x(0) - b(0)\right)M(0) \\
&- 2a(0)\left(m_1M(0) + (m_2-1)M_x(0)\right), \\
T_4 =&(m_3-1)(a_x(0)-b(0))K_2(0,x) \\
&- 2a(0)\left[(m_2-1)K_{2,x}(0,x) + m_1K_2(0,x)\right], \\
T_5(x) =& -2m_2(m_3-1)a(0)K_2(0,x), \\
T_6(x) =& 2(m_3-1)K_2(0,x), \\
T_7 =& - a_x(1)M(1) + a(1)M_x(1) + b(1)M(1), \\
T_8 =& 2a(1)M(1),
\end{aligned}
$$

where $K_{1,x}(1,x) = \left[K_{1,x}(x,\xi)|_{x=1}\right]_{\xi=x}$, $K_{2,x}(0,x) = \left[K_{2,x}(x,\xi)|_{x=0}\right]_{\xi=x}$ *and* $\epsilon > 0$ *and* m_i, $i \in \{1,\cdots,3\}$, *are scalars.*

Lemma B.7. *Suppose we are given* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$,

$$
\{T_0, T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\} = \mathcal{N}(M, K_1, K_2),
$$

and scalars m_i, $i \in \{1,\cdots,3\}$, *as defined in Definition 6.2. Then, for the solution* $w(x,t)$ *of Equations* (6.1)-(6.2) *or Equations* (6.21)-(6.22), $\mathcal{A}$ *as defined in Equation* (6.7) *and* $\mathcal{P}$ *defined in Equation* (5.12), *we have that*

$$
\begin{aligned}
&\langle \mathcal{A}\mathcal{P}z(\cdot,t), z(\cdot,t)\rangle + \langle z(\cdot,t), \mathcal{P}\mathcal{A}z(\cdot,t)\rangle \\
&\le \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle \\
&\quad + z(0,t)\left(T_3z(0,t) + \int_0^1 T_4(x)z(x,t)dx\right) + z_x(0,t)\int_0^1 T_5(x)z(x,t)dx \\
&\quad + \int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2\right)z(0,t)
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^1 \frac{1}{M(0)} T_6(x) z(x,t) dx \int_0^1 \alpha \epsilon \pi^2 z(x,t) dx \\
&+ z(1,t) \left(T_7 z(1,t) + T_8 z_x(1,t) \right),
\end{aligned}
$$

where $z(\cdot, t) = \mathcal{P}^{-1} w(\cdot, t)$, *and* $\mathcal{T}$ *is defined as*

$$
(\mathcal{T} y)(x) = T_0(x) y(x) + \int_0^x T_1(x,\xi) y(\xi) d\xi + \int_x^1 T_2(x,\xi) y(\xi) d\xi, \quad y \in L_2(0,1).
$$

Proof. We begin by considering the following decomposition

$$
\begin{aligned}
&\langle \mathcal{A}\mathcal{P} z(\cdot,t), z(\cdot,t) \rangle + \langle z(\cdot,t), \mathcal{A}\mathcal{P} z(\cdot,t) \rangle \\
&= 2 \int_0^1 \left(a(x) \frac{\partial^2}{\partial x^2} (\mathcal{P} z)(x,t) + b(x) \frac{\partial}{\partial x} (\mathcal{P} z)(x,t) + c(x) (\mathcal{P} z)(x,t) \right) z(x,t) dx \\
&= 2 \left(\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \right),
\end{aligned} \tag{B.8}
$$

where

$$
\begin{aligned}
\Gamma_1 &= \int_0^1 z_{xx}(x,t) \left[a(x) M(x) \right] z(x,t) dx, \\
\Gamma_2 &= \int_0^1 z_x(x,t) \left[2a(x) M_x(x) + b(x) M(x) \right] z(x,t) dx, \\
\Gamma_3 &= \int_0^1 z^2(x,t) \left[a(x) \left(M_{xx}(x) + K_{1,x}(x,x) - K_{2,x}(x,x) \right) + b(x) M_x(x) \right] dx \\
&\quad + \int_0^1 z^2(x,t) M(x) c(x) dx, \\
\Gamma_4 &= \int_0^1 \int_0^x z(x,t) \left[a(x) K_{1,xx}(x,\xi) + b(x) K_{1,x}(x,\xi) + c(x) K_1(x,\xi) \right] z(\xi,t) d\xi dx \\
&\quad + \int_0^1 \int_x^1 z(x,t) \left[a(x) K_{2,xx}(x,\xi) + b(x) K_{2,x}(x,\xi) + c(x) K_2(x,\xi) \right] z(\xi,t) d\xi dx.
\end{aligned}
$$

Applying integration by parts twice

$$
\begin{aligned}
\Gamma_1 = &- \int_0^1 z_x^2(x,t) a(x) M(x) dx + \int_0^1 z^2(x,t) \frac{1}{2} \frac{\partial^2}{\partial x^2} \left(a(x) M(x) \right) dx \\
&+ z(1,t) \left(-\frac{1}{2} \left[a_x(1) M(1) + a(1) M_x(1) \right] z(1,t) + a(1) M(1) z_x(1,t) \right) \\
&+ z(0,t) \left(\frac{1}{2} \left[a_x(0) M(0) + a(0) M_x(0) \right] z(0,t) - a(0) M(0) z_x(0,t) \right).
\end{aligned}
$$

Applying a variation of Wirtinger's inequality

$$\begin{aligned}\Gamma_1 \leq &\int_0^1 z^2(x,t)\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}\left(a(x)M(x)\right) - \frac{\alpha\epsilon\pi^2}{2}\right)dx\\ &+ z(1,t)\left(-\frac{1}{2}\left[a_x(1)M(1) + a(1)M_x(1)\right]z(1,t) + a(1)M(1)z_x(1,t)\right)\\ &+ z(0,t)\left(\frac{1}{2}\left[a_x(0)M(0) + a(0)M_x(0) + \frac{\alpha\epsilon\pi^2}{2}\right]z(0,t) - a(0)M(0)z_x(0,t)\right).\end{aligned} \tag{B.9}$$

Applying integration by parts

$$\begin{aligned}\Gamma_2 = &-\int_0^1 z^2(x,t)\left(a_x(x)M_x(x) + a(x)M_{xx}(x) + \frac{1}{2}\frac{\partial}{\partial x}\left(b(x)M(x)\right)\right)dx\\ &+ z^2(1,t)\left(a(1)M_x(1) + \frac{1}{2}b(1)M(1)\right) - z^2(0,t)\left(a(0)M_x(0) + \frac{1}{2}b(0)M(0)\right).\end{aligned} \tag{B.10}$$

Adding Equations (B.9) and (B.10)

$$\begin{aligned}&\Gamma_1 + \Gamma_2\\ &\leq \int_0^1 z^2(x,t)\left(\frac{1}{2}a_{xx}(x)M(x) - \frac{1}{2}a(x)M_{xx}(x) - \frac{1}{2}b_x(x)M(x) - \frac{1}{2}b(x)M_x(x)\right)dx\\ &\quad - \int_0^1 \frac{\alpha\epsilon\pi^2}{4}z^2(x,t)dx + z(1,t)\left(\frac{1}{2}T_7 z(1,t) + \frac{1}{2}T_8 z_x(1,t)\right)\\ &\quad + \left(-\frac{1}{2}a(0)M_x(0) + \frac{1}{4}\alpha\epsilon\pi^2\right)z(0,t)^2\\ &\quad + z(0,t)\left(\frac{1}{2}a_x(0) - \frac{1}{2}b(0)\right)M(0)z(0,t) - z(0,t)a(0)M(0)z_x(0,t)\\ &\quad - a(0)M_x(0)z(0,t)^2.\end{aligned} \tag{B.11}$$

Since $z(\cdot,t) = \mathcal{P}^{-1}w(\cdot,t)$, $w(\cdot,t) = \mathcal{P}z(\cdot,t)$. Thus

$$\begin{aligned}2w(x,t) =& M(x)z(x,t) + \int_0^x K_1(x,\xi)z(\xi,t)d\xi + \int_x^1 K_2(x,\xi)z(\xi,t)d\xi \text{ and}\\ w_x(x,t) =& M_x(x)z(x,t) + M(x)z_x(x,t) + \int_0^x K_{1,x}(x,\xi)z(\xi,t)d\xi\\ &+ \int_x^1 K_{2,x}(x,\xi)z(\xi,t)d\xi.\end{aligned}$$

The boundary condition for $x = 0$ can hence be written as

$$\begin{aligned} w(0,t) =& M(0)z(0,t) + \int_0^1 K_2(0,x)z(x,t)dx, \\ w_x(0,t) =& M_x(0)z(0,t) + M(0)z_x(0,t) + \int_0^1 K_{2,x}(0,x)z(x,t)dx. \end{aligned}$$

Using Definition 6.2,

$$w_x(0,t) = m_1 w(0,t) + m_2 w_x(0,t), \quad w(0,t) = m_3 w(0,t),$$

the boundary conditions in variable z can be written as

$$z(0,t) = m_3 z(0,t) + \int_0^1 (m_3 - 1)\frac{1}{M(0)} K_2(0,x)z(x,t)dx, \tag{B.12}$$

$$M(0)z(0,t) = m_3 M(0)z(0,t) + \int_0^1 (m_3 - 1)K_2(0,x)z(x,t)dx, \tag{B.13}$$

$$\begin{aligned} M(0)z_x(0,t) =& \left[m_1 M(0) + (m_2 - 1)M_x(0)\right] z(0,t) + m_2 M(0) z_x(0,t) \\ &+ \int_0^1 \left[(m_2 - 1)K_{2,x}(0,x) + m_1 K_2(0,x)\right] z(x,t)dx. \end{aligned} \tag{B.14}$$

Substituting Equations (B.12)-(B.14) in Equation (B.11) produces

$$\begin{aligned} &\Gamma_1 + \Gamma_2 \\ &\leq \int_0^1 z^2(x,t) \left(\frac{1}{2}a_{xx}(x)M(x) - \frac{1}{2}a(x)M_{xx}(x) - \frac{1}{2}b_x(x)M(x) - \frac{1}{2}b(x)M_x(x)\right) dx \\ &\quad - \int_0^1 \frac{\pi^2}{4}\alpha\epsilon z^2(x,t)dx + z(0,t)\frac{1}{2}\left(T_3 z(0,t) + \int_0^1 T_4(x)z(x,t)dx\right) \\ &\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx \left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2\right) z(0,t) \\ &\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx \int_0^1 \alpha\epsilon\pi^2 z(x,t)dx \\ &\quad - m_2 a(0) z_x(0,t) M(0) z(0,t) + z(1,t)\left(\frac{1}{2}T_7 z(1,t) + \frac{1}{2}T_8 z_x(1,t)\right). \end{aligned}$$

Substituting the boundary condition in Equation (B.13) in the second to last term of the previous equation we obtain

$$\Gamma_1 + \Gamma_2$$

$$
\begin{aligned}
&\leq \int_0^1 z^2(x,t)\left(\frac{1}{2}a_{xx}(x)M(x) - \frac{1}{2}a(x)M_{xx}(x) - \frac{1}{2}b_x(x)M(x) - \frac{1}{2}b(x)M_x(x)\right)dx \\
&\quad - \int_0^1 \frac{\pi^2}{4}\alpha\epsilon z^2(x,t)dx + z(0,t)\frac{1}{2}\left(T_3 z(0,t) + \int_0^1 T_4(x)z(x,t)dx\right) \\
&\quad + z_x(0,t)\frac{1}{2}\int_0^1 T_5(x)z(x,t)dx \\
&\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2\right)z(0,t) \\
&\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\int_0^1 \alpha\epsilon\pi^2 z(x,t)dx \\
&\quad + z(1,t)\left(\frac{1}{2}T_7 z(1,t) + \frac{1}{2}T_8 z_x(1,t)\right) - z(0,t)m_2 m_3 M(0)z_x(0,t).
\end{aligned}
$$

Recall from Definition 6.2 that for all possible cases, $m_2 m_3 = 0$. Thus,

$$
\begin{aligned}
&\Gamma_1 + \Gamma_2 \\
&\leq \int_0^1 z^2(x,t)\left(\frac{1}{2}a_{xx}(x)M(x) - \frac{1}{2}a(x)M_{xx}(x) - \frac{1}{2}b_x(x)M(x) - \frac{1}{2}b(x)M_x(x)\right)dx \\
&\quad - \int_0^1 \frac{\pi^2}{4}\alpha\epsilon z^2(x,t)dx + z(0,t)\frac{1}{2}\left(T_3 z(0,t) + \int_0^1 T_4(x)z(x,t)dx\right) \\
&\quad + z_x(0,t)\frac{1}{2}\int_0^1 T_5(x)z(x,t)dx \\
&\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2\right)z(0,t) \\
&\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\int_0^1 \alpha\epsilon\pi^2 z(x,t)dx \\
&\quad + z(1,t)\left(\frac{1}{2}T_7 z(1,t) + \frac{1}{2}T_8 z_x(1,t)\right).
\end{aligned}
\tag{B.15}
$$

Adding Equation (B.15) and Γ_3 produces

$$
\begin{aligned}
&\Gamma_1 + \Gamma_2 + \Gamma_3 \\
&\leq \int_0^1 z^2(x,t)\frac{1}{2}T_0(x)dx \\
&\quad + z(0,t)\frac{1}{2}\left(T_3 z(0,t) + \int_0^1 T_4(x)z(x,t)dx\right) + z_x(0,t)\frac{1}{2}\int_0^1 T_5(x)z(x,t)dx \\
&\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2\right)z(0,t) \\
&\quad + \frac{1}{2}\int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\int_0^1 \alpha\epsilon\pi^2 z(x,t)dx
\end{aligned}
$$

$$+\, z(1,t)\left(\frac{1}{2}T_7 z(1,t) + \frac{1}{2}T_8 z_x(1,t)\right). \tag{B.16}$$

Switching the order of integration and interchanging x and ξ produces

$$\Gamma_4 = \int_0^1 \int_0^x z(x,t)\frac{1}{2}T_2(x,\xi)z(\xi,t)d\xi + \int_0^1 \int_x^1 z(x,t)\frac{1}{2}T_3(x,\xi)z(\xi,t)d\xi. \tag{B.17}$$

Finally, substituting Equations (B.16)-(B.17) into Equation (B.8) produces

$$\begin{aligned}
&\langle \mathcal{A}\mathcal{P}z(\cdot,t), z(\cdot,t)\rangle + \langle z(\cdot,t), \mathcal{P}\mathcal{A}z(\cdot,t)\rangle \\
&\leq \langle z(\cdot,t), \mathcal{T}z(\cdot,t)\rangle \\
&\quad + z(0,t)\left(T_3 z(0,t) + \int_0^1 T_4(x)z(x,t)dx\right) + z_x(0,t)\int_0^1 T_5(x)z(x,t)dx \\
&\quad + \int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\left(-a(0)M_x(0) + \frac{1}{2}\alpha\epsilon\pi^2\right)z(0,t) \\
&\quad + \int_0^1 \frac{1}{M(0)}T_6(x)z(x,t)dx\int_0^1 \alpha\epsilon\pi^2 z(x,t)dx \\
&\quad + z(1,t)\left(T_7 z(1,t) + T_8 z_x(1,t)\right).
\end{aligned}$$

$\square$

APPENDIX C

POSITIVE OPERATORS AND THEIR INVERSES

Proof of Theorem 5.5. By non-negativity, there exists a $\bar{U}$ such that $U = \bar{U}^T\bar{U}$. Partitioning $\bar{U}$ as

$$\bar{U} = \begin{bmatrix} D & H_1 & H_2 \end{bmatrix}$$

gives us

$$U = \begin{bmatrix} D^TD & D^TH_1 & D^TH_2 \\ H_1^TD & H_1^TH_1 & H_1^TH_2 \\ H_2^TD & H_2^TH_1 & H_2^TH_2 \end{bmatrix} = \begin{bmatrix} U_{11} - \epsilon I_0 & U_{12} & U_{13} \\ \star & U_{22} & U_{23} \\ \star & \star & U_{33} \end{bmatrix} \tag{C.1}$$

Let, for $y \in L_2(0,1)$,

$$(Ay)(\eta) = DZ_1(\eta)y(\eta) + \int_0^\eta H_1Z_2(\eta,x)y(x)dx + \int_\eta^1 H_2Z_2(\eta,x)y(x)dx.$$

Similarly,

$$(Ay)(\eta) = DZ_1(\eta)y(\eta) + \int_0^\eta H_1Z_2(\eta,\xi)y(\xi)d\xi + \int_\eta^1 H_2Z_2(\eta,\xi)y(\xi)d\xi.$$

Thus,

$$\begin{aligned}
&\langle Ay, Ay\rangle \\
&= \int_0^1 \left(y(\eta)^TZ_1(\eta)^TD^T + \int_0^\eta y(x)^TZ_2(\eta,x)^TH_1^Tdx + \int_\eta^1 y(x)^TZ_2(\eta,x)^TH_2^Tdx\right) \\
&\qquad \left(DZ_1(\eta)y(\eta) + \int_0^\eta H_1Z_2(\eta,\xi)y(\xi)d\xi + \int_\eta^1 H_2Z_2(\eta,\xi)y(\xi)d\xi\right)d\eta \\
&= A_1 + A_2 + A_3,
\end{aligned} \tag{C.2}$$

where

$$\begin{aligned}
A_1 &= \int_0^1 y(\eta)^TZ_1(\eta)^T\left(U_{11} - \epsilon I_0\right)Z_1(\eta)y(\eta)d\eta \\
&\quad + \int_0^1 y(\eta)^TZ_1(\eta)^T\left(\int_0^\eta U_{12}Z_2(\eta,\xi)y(\xi)d\xi + \int_\eta^1 U_{13}Z_2(\eta,\xi)y(\xi)d\xi\right)d\eta, \\
A_2 &= \int_0^1 \left(\int_0^\eta y(x)^TZ_2(\eta,x)^TU_{21}dx + \int_\eta^1 y(x)^TZ_2(\eta,x)^TU_{31}dx\right)Z_1(\eta)y(\eta)d\eta
\end{aligned}$$

and

$$A_3 =$$

$$\int_0^1 \int_0^\eta y(x)^T Z_2(\eta,x)^T \left(U_{22} \int_0^\eta Z_2(\eta,\xi) y(\xi) d\xi + U_{23} \int_\eta^1 Z_2(\eta,\xi) y(\xi) d\xi \right) dx d\eta$$

$$+ \int_0^1 \int_\eta^1 y(x)^T Z_2(\eta,x)^T \left(U_{32} \int_0^\eta Z_2(\eta,\xi) y(\xi) d\xi + U_{33} \int_\eta^1 Z_2(\eta,\xi) y(\xi) d\xi \right) dx d\eta.$$

Note that here we have used the definitions of U_{ij}.

Switching between η and x in A_1

$$\begin{aligned} A_1 = & \int_0^1 y(x)^T Z_1(x)^T \left(U_{11} - \epsilon I\right) Z_1(x) y(x) dx \\ & + \int_0^1 \int_0^x y(x)^T Z_1(x)^T U_{12} Z_2(x,\xi) y(\xi) d\xi dx \\ & + \int_0^1 \int_x^1 y(x)^T Z_1(x)^T U_{13} Z_2(x,\xi) y(\xi) d\xi dx. \end{aligned} \tag{C.3}$$

Switching between η and ξ and switching the order of integration in A_2

$$A_2 = \int_0^1 y(x)^T \left(\int_0^x Z_2(\xi,x)^T U_{31} Z_1(\xi) y(\xi) d\xi + \int_x^1 Z_2(\xi,x)^T U_{21} Z_1(\xi) y(\xi) d\xi \right) dx \tag{C.4}$$

Switching the order of integration, first between x and η and then between ξ and η in A_3, we get

$$\begin{aligned} A_3 = & \int_0^1 y(x)^T \int_0^x \left(\int_0^\xi Z_2(\eta,x)^T U_{33} Z_2(\eta,\xi) d\eta + \int_\xi^x Z_2(\eta,x)^T U_{32} Z_2(\eta,\xi) d\eta \right. \\ & \left. + \int_x^1 Z_2(\eta,x)^T U_{22} Z_2(\eta,\xi) d\eta \right) y(\xi) d\xi dx \\ & + \int_0^1 y(x)^T \int_x^1 \left(\int_0^x Z_2(\eta,x)^T U_{33} Z_2(\eta,\xi) d\eta + \int_x^\xi Z_2(\eta,x)^T U_{23} Z_2(\eta,\xi) d\eta \right. \\ & \left. + \int_\xi^1 Z_2(\eta,x)^T U_{22} Z_2(\eta,\xi) d\eta \right) y(\xi) d\xi dx. \end{aligned} \tag{C.5}$$

Substituting Equations (C.3)-(C.5) into (C.2) and using the definitions of K_1 and K_2 gives

$$\begin{aligned}
&\langle Ay, Ay\rangle \\
&= \int_0^1 y(x)\Bigg(\left[Z_1(x)^T U_{11} Z_1(x) - \epsilon Z_1(x)^T I_0 Z_1(x)\right] y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \int_x^1 K_2(x,\xi)y(\xi)d\xi \Bigg) dx.
\end{aligned}$$

From the theorem statement, $M(x) \geq Z_1(x)^T U_{11} Z_1(x)$. Therefore,

$$\begin{aligned}
&\langle Ay, Ay\rangle \\
&\leq \int_0^1 y(x)\Bigg(\left[M(x) - \epsilon Z_1(x)^T I_0 Z_1(x)\right] y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \int_x^1 K_2(x,\xi)y(\xi)d\xi \Bigg) dx \\
&= \langle y, \mathcal{P}y\rangle - \epsilon \int_0^1 y(x) Z_1(x)^T I_0 Z_1(x) y(x) dx.
\end{aligned}$$

Since $\langle Ay, Ay\rangle \geq 0$, using the previous expression we get that

$$\langle y, \mathcal{P}y\rangle - \epsilon \int_0^1 y(x) Z_1(x)^T I_0 Z_1(x) y(x) dx \geq 0.$$

Finally, since $Z_1(x)^T I_0 Z_1(x) = 1$, we obtain

$$\langle y, \mathcal{P}y\rangle - \epsilon \int_0^1 y(x) Z_1(x)^T I_0 Z_1(x) y(x) dx = \langle y, \mathcal{P}y\rangle - \epsilon \|y\|^2 \geq 0.$$

Therefore

$$\langle y, \mathcal{P}y\rangle \geq \epsilon \|y\|^2, \quad \text{for all} \quad y \in L_2(0,1).$$

Self-adjointedness of $\mathcal{P}$ can be established using the fact that by construction $K_1(x,\xi) = K_2(\xi, x)$. $\square$

Lemma C.1. *Let* $\{M, K_1, K_2\} = \Omega_{d_1,d_2,\epsilon_1,\epsilon_2}$ *for any* $0 < \epsilon_1 < \epsilon_2$. *Then for the following operator*

$$(\mathcal{P}y)(x) = M(x)y(x) + \int_0^x K_1(x,\xi)y(\xi)d\xi + \int_x^1 K_2(x,\xi)y(\xi)d\xi, \quad y \in L_2(0,1),$$

the following holds

$$\frac{1}{\epsilon_2}\|y\|^2 \leq \left\langle y, \mathcal{P}^{-1}y \right\rangle \leq \frac{1}{\epsilon_1}\|y\|^2.$$

Proof. Since $\{M, K_1, K_2\} = \Omega_{d_1,d_2,\epsilon_1,\epsilon_2}$, from Corollary 5.6 we have that

$$\epsilon_1\|y\|^2 \leq \langle y, \mathcal{P}y\rangle \leq \epsilon_2\|y\|^2.$$

Now,

$$\langle y, \mathcal{P}y\rangle \leq \epsilon_2\|y\|^2 = \epsilon_2 \langle y, y\rangle.$$

Thus,

$$\langle y, (\mathcal{P} - \epsilon_2\mathcal{I})\, y\rangle \leq 0,$$

where $\mathcal{I}$ is the identity operator. From Theorem 6.9, we know that the inverse of theis operator $\mathcal{P}^{-1}$ exists. Thus,

$$\left\langle y, \mathcal{P}\left(\mathcal{I} - \epsilon_2\mathcal{P}^{-1}\right) y\right\rangle \leq 0.$$

By definition $\mathcal{P}$ is a positive operator. Thus, by [35, 9.4-2], $\mathcal{P}$ has a unique positive self-adjoint square root, that is,

$$\mathcal{P} = \mathcal{P}^{\frac{1}{2}}\mathcal{P}^{\frac{1}{2}}.$$

Thus, we get

$$\left\langle y, \mathcal{P}^{\frac{1}{2}}\mathcal{P}^{\frac{1}{2}}\left(\mathcal{I} - \epsilon_2\mathcal{P}^{-1}\right) y\right\rangle \leq 0.$$

Since $\mathcal{P}^{\frac{1}{2}}$ is self-adjoint

$$\left\langle \mathcal{P}^{\frac{1}{2}}y, \mathcal{P}^{\frac{1}{2}}\left(\mathcal{I} - \epsilon_2\mathcal{P}^{-1}\right) y\right\rangle \leq 0.$$

Using [35, 9.4-2] we get that since $\mathcal{P}$ commutes with $\mathcal{P}^{-1}$, $\mathcal{P}^{\frac{1}{2}}$ commutes with $\mathcal{P}^{-1}$. Therefore

$$\left\langle \mathcal{P}^{\frac{1}{2}} y, \mathcal{P}^{\frac{1}{2}} \left(\mathcal{I} - \epsilon_2 \mathcal{P}^{-1}\right) y \right\rangle = \left\langle \mathcal{P}^{\frac{1}{2}} y, \left(\mathcal{I} - \epsilon_2 \mathcal{P}^{-1}\right) \mathcal{P}^{\frac{1}{2}} y \right\rangle \le 0.$$

Thus, we conclude that

$$\mathcal{I} - \epsilon_2 \mathcal{P}^{-1} \le 0, \text{ on } L_2(0,1).$$

Therefore, for any $y \in L_2(0,1)$, we have that

$$\left\langle y, \left(\mathcal{I} - \epsilon_2 \mathcal{P}^{-1}\right) y \right\rangle \le 0.$$

This implies that, for any $y \in L_2(0,1)$,

$$\frac{1}{\epsilon_2} \|y\|^2 \le \left\langle y, \mathcal{P}^{-1} y \right\rangle.$$

The assertion that

$$\left\langle y, \mathcal{P}^{-1} y \right\rangle \le \frac{1}{\epsilon_1} \|y\|^2,$$

is similarly proved.

□

Proof of Lemma 6.8. Let $\|\cdot\|_{\mathbb{R}^{k\times k}}$ be any induced norm on $\mathbb{R}^{k\times k}$. Then, for any matrix valued function $Q : [0,1] \to \mathbb{R}^{k\times k}$ define

$$\|Q\|_\infty = \sup_{x\in[0,1]} \|Q(x)\|_{\mathbb{R}^{k\times k}}.$$

It can be easily verified that the space

$$\Phi = \{Q : [0,1] \to \mathbb{R}^{k\times k} : \quad \|Q\|_\infty < \infty\},$$

where $\|\cdot\|_\infty$ is the norm, is a complete normed space. In other words, the space Φ with norm $\|\cdot\|_\infty$ is a Banach space.

For any $V \in \Phi$, we define the following mapping

$$(TV)(x) = I + \int_0^x A(\xi)V(\xi)d\xi.$$

Then for any $V, W \in \Phi$,

$$(TV)(x) - (TW)(x) = \int_0^x A(\xi)\left[V(\xi) - W(\xi)\right]d\xi.$$

Thus,

$$\begin{aligned} \|(TV)(x) - (TW)(x)\|_{\mathbb{R}^{k\times k}} &= \left\| \int_0^x A(\xi)\left[V(\xi) - W(\xi)\right]d\xi \right\|_{\mathbb{R}^{k\times k}} \\ &\leq \int_0^x \|A(\xi)\|_{\mathbb{R}^{k\times k}} \|V(\xi) - W(\xi)\|_{\mathbb{R}^{k\times k}} d\xi. \end{aligned} \tag{C.6}$$

Since the elements of $A(x)$ are continuous on $[0, 1]$, $A \in \Phi$. Let $\alpha = \|A\|_\infty$, then

$$\|A(\xi)\|_{\mathbb{R}^{k\times k}} \leq \alpha, \quad \text{for all} \quad \xi \in [0, 1].$$

Moreover,

$$\|V(\xi) - W(\xi)\|_{\mathbb{R}^{k\times k}} \leq \|V - W\|_\infty, \quad \text{for all} \quad \xi \in [0, 1].$$

Thus, substituting these in Equation (C.6) produces

$$\begin{aligned} \|(TV)(x) - (TW)(x)\|_{\mathbb{R}^{k\times k}} &\leq \alpha\|V - W\|_\infty \int_0^x d\xi \\ &= \alpha\|V - W\|_\infty x, \quad \text{for all} \quad x \in [0, 1]. \end{aligned} \tag{C.7}$$

We will now prove that for any $m \in \mathbb{N}$, the following holds

$$\|(T^m V)(x) - (T^m W)(x)\|_{\mathbb{R}^{k\times k}} \leq \frac{\alpha^m x^m}{m!}\|V - W\|_\infty. \tag{C.8}$$

Clearly, from Equation (C.7), this claim is true for $m = 1$. Assume that Equation (C.8) holds for any $m \in \mathbb{N}$. Then

$$\begin{aligned} &\|(T^{m+1}V)(x) - (T^{m+1}W)(x)\|_{\mathbb{R}^{k\times k}} \\ &= \left\| \int_0^x A(\xi)\left[(T^m V)(\xi) - (T^m W)(\xi)\right]d\xi \right\|_{\mathbb{R}^{k\times k}} \end{aligned}$$

$$\leq \int_0^x \|A(\xi)\|_{\mathbb{R}^{k\times k}} \| [(T^m V)(\xi) - (T^m W)(\xi)\|]_{\mathbb{R}^{k\times k}} \, d\xi$$
$$\leq \alpha \int_0^x \| [(T^m V)(\xi) - (T^m W)(\xi)\|]_{\mathbb{R}^{k\times k}} \, d\xi.$$

Substituting in Equation (C.8) produces

$$\begin{aligned}\|(T^{m+1}V)(x) - (T^{m+1}W)(x)\|_{\mathbb{R}^{k\times k}} \leq & \alpha \|V - W\|_\infty \int_0^x \frac{\alpha^m \xi^m}{m!} d\xi \\ = & \frac{\alpha^m x^m}{m!} \|V - W\|_\infty.\end{aligned}$$

Thus, we have proven by induction that

$$\begin{aligned}\|(T^m V)(x) - (T^m W)(x)\|_{\mathbb{R}^{k\times k}} \leq & \frac{\alpha^m x^m}{m!} \|V - W\|_\infty \\ \leq & \frac{\alpha^m}{m!} \|V - W\|_\infty, \quad \text{for all} \quad x \in [0, 1].\end{aligned}$$

Since

$$\|T^m V - T^m W\|_\infty = \sup_{x\in[0,1]} \|(T^m V)(x) - (T^m W)(x)\|_{\mathbb{R}^{k\times k}},$$

we conclude

$$\|T^m V - T^m W\|_\infty \leq \frac{\alpha^m}{m!} \|V - W\|_\infty.$$

Since $V, W \in \Phi$ were chosen arbitrarily, and for a large enough $m \in \mathbb{N}$

$$\frac{\alpha^m}{m!} < 1,$$

we conclude that T^m, for a large enough $m \in \mathbb{N}$, is a contraction on Φ [35, 5.1-1]. Therefore, from Banach fixed point theorem [35, 5.1-2], there exists a unique fixed point $U \in \Phi$ which satisfies

$$U = T^m U,$$

and U can be obtained by the uniform limit of

$$U_0 = I, \quad U_1 = T^m U_0, \quad U_2 = T^{2m} U_1, \cdots, U_n = T^{nm} U_{n-1}, \cdots .$$

Moreover, from [35, Lemma 5.4-3], $U \in \Phi$ is also the unique solution to

$$U = TU$$

and hence is given by the uniform limit of the sequence

$$U_0 = I, \quad U_1 = TU_0, \quad U_2 = T^2U_1, \cdots, U_n = T^nU_{n-1}, \cdots.$$

Since the unique fixed point U satisfies $U = TU$, using the definition of the mapping T,

$$U(x) = I + \int_0^x A(\xi)U(\xi)d\xi.$$

Thus, by differentiating in x, we see that the fixed point U satisfies

$$\frac{dU(x)}{dx} = A(x)U(x)$$

and

$$U(0) = I.$$

To prove that $U(x)$ is non-singular for every $x \in [0, 1]$, one may apply the small-gain theorem [52, 3.7] and use the fact that $U(x)$ is the uniform limit of the sequence $U_n(x)$ provided previously.

$\square$

Proof of Theorem 6.9. We begin by noting that $U(x)$, the fundamental matrix of $-B(x)M(x)^{-1}C(x)$, exists and is non-singular for all $x \in [0, 1]$. This is due to the fact that the elements of the matrix $-B(x)M(x)^{-1}C(x)$ are rational functions, and hence, Lebesgue integrable. Thus, by Lemma 6.8, the result follows.

The integral kernels γ_1 and γ_2 are well defined if the matrix P is well-defined. Let

$$U(1) = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

Then

$$N_1 + N_2U(1) = \begin{bmatrix} I_{2(d+1)} & 0 \\ \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \\ 0 & I_{2(d+1)} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} I_{2(d+1)} & 0 \\ \\ U_{21} & U_{22} \end{bmatrix}.$$

Thus

$$(N_1 + N_2U(1))^{-1} = \begin{bmatrix} I_{2(d+1)} & 0 \\ \\ -U_{22}^{-1}U_{21} & U_{22}^{-1} \end{bmatrix}.$$

Since $U(x)$ is invertible, so is $U(1)$. Hence U_{22}^{-1} exists and consequently, $(N_1 + N_2U(1))^{-1}$ is well defined. Thus, the matrix $P = (N_1 + N_2U(1))^{-1} N_2U(1)$ and the integral kernels γ_1 and γ_2 are well defined.

Since $\Theta_{\mathcal{P}} = (M, F_1, F_2, G_1, G_2)$, we have that

$$\begin{aligned}(\mathcal{P}w)(x) =& M(x)w(x) + \int_0^x K_1(x,\xi)w(\xi)d\xi + \int_x^1 K_2(x,\xi)w(\xi)d\xi \\ =& M(x)w(x) + \int_0^x F_1(x)G_1(\xi)w(\xi)d\xi - \int_x^1 F_2(x)G_2(\xi)w(\xi)dxi.\end{aligned}$$

It can be easily established that

$$K_1(x,\xi) = F_1(x)G_1(\xi) = C(x)N_1B(\xi), \quad K_2(x,\xi) = \ -F_2(x)G_2(\xi) = -C(x)N_2B(\xi). \tag{C.9}$$

Now, from the theorem hypothesis, we have that

$$\left(\hat{\mathcal{P}}w\right)(x) = M(x)^{-1}w(x) - \int_0^x \gamma_1(x,\xi)w(\xi)d\xi - \int_x^1 \gamma_2(x,\xi)w(\xi)d\xi.$$

Then,

$$\begin{aligned}&\left(\mathcal{P}\hat{\mathcal{P}}w\right)(x) \\ &= M(x)\left(\hat{\mathcal{P}}w\right)(x) + \int_0^x K_1(x,\xi)\left(\hat{\mathcal{P}}w\right)(\xi)d\xi + \int_x^1 K_2(x,\xi)\left(\hat{\mathcal{P}}w\right)(\xi)d\xi \\ &= M(x)\left(M(x)^{-1}w(x) - \int_0^x \gamma_1(x,\xi)w(\xi)d\xi - \int_x^1 \gamma_2(x,\xi)w(\xi)d\xi\right)\end{aligned}$$

$$+\int_0^x K_1(x,\xi)\left(M(\xi)^{-1}w(\xi)-\int_0^\xi \gamma_1(\xi,\theta)w(\theta)d\theta-\int_\xi^1 \gamma_2(\xi,\theta)w(\theta)d\theta\right)d\xi$$
$$+\int_x^1 K_2(x,\xi)\left(M(\xi)^{-1}w(\xi)-\int_0^\xi \gamma_1(\xi,\theta)w(\theta)d\theta-\int_\xi^1 \gamma_2(\xi,\theta)w(\theta)d\theta\right)d\xi.$$

Thus,

$$\begin{aligned}
&\left(\mathcal{P}\hat{\mathcal{P}}w\right)(x)\\
&= w(x)+\int_0^x \left(-M(x)\gamma_1(x,\xi)+K_1(x,\xi)M(\xi)^{-1}\right)w(\xi)d\xi\\
&\quad+\int_x^1 \left(-M(x)\gamma_2(x,\xi)+K_2(x,\xi)M(\xi)^{-1}\right)w(\xi)d\xi\\
&\quad-\int_0^x\int_0^\xi K_1(x,\xi)\gamma_1(x,\xi)w(\theta)d\theta d\xi-\int_0^x\int_\xi^1 K_1(x,\xi)\gamma_2(\xi,\theta)w(\theta)d\theta d\xi\\
&\quad-\int_x^1\int_0^\xi K_2(x,\xi)\gamma_1(\xi,\theta)w(\theta)d\theta d\xi-\int_x^1\int_\xi^1 K_2(x,\xi)\gamma_2(\xi,\theta)w(\theta)d\theta d\xi.
\end{aligned}$$

Changing the order of integration in the last four integrals

$$\begin{aligned}
&\left(\mathcal{P}\hat{\mathcal{P}}w\right)(x)\\
&= w(x)+\int_0^x \left(-M(x)\gamma_1(x,\xi)+K_1(x,\xi)M(\xi)^{-1}\right)w(\xi)d\xi\\
&\quad+\int_x^1 \left(-M(x)\gamma_2(x,\xi)+K_2(x,\xi)M(\xi)^{-1}\right)w(\xi)d\xi\\
&\quad-\int_0^x\int_\theta^x K_1(x,\xi)\gamma_1(\xi,\theta)d\xi w(\theta)d\theta-\int_0^x\int_0^\theta K_1(x,\xi)\gamma_2(\xi,\theta)d\xi w(\theta)d\theta\\
&\quad-\int_x^1\int_0^x K_1(x,\xi)\gamma_2(\xi,\theta)d\xi w(\theta)d\theta-\int_0^x\int_x^1 K_2(x,\xi)\gamma_1(\xi,\theta)d\xi w(\theta)d\theta\\
&\quad-\int_x^1\int_\theta^1 K_2(x,\xi)\gamma_1(\xi,\theta)d\xi w(\theta)d\theta-\int_x^1\int_x^\theta K_2(x,\xi)\gamma_2(\xi,\theta)d\xi w(\theta)d\theta.
\end{aligned}$$

Switching between θ and ξ in the last six integrals produces

$$\begin{aligned}
&\left(\mathcal{P}\hat{\mathcal{P}}w\right)(x)\\
&= w(x)+\int_0^x \left(-M(x)\gamma_1(x,\xi)+K_1(x,\xi)M(\xi)^{-1}\right)w(\xi)d\xi\\
&\quad+\int_x^1 \left(-M(x)\gamma_2(x,\xi)+K_2(x,\xi)M(\xi)^{-1}\right)w(\xi)d\xi\\
&\quad-\int_0^x\int_\xi^x K_1(x,\theta)\gamma_1(\theta,\xi)d\theta w(\xi)d\xi-\int_0^x\int_0^\xi K_1(x,\theta)\gamma_2(\theta,\xi)d\theta w(\xi)d\xi
\end{aligned}$$

$$- \int_x^1 \int_0^x K_1(x,\theta)\gamma_2(\theta,\xi)d\theta w(\xi)d\xi - \int_0^x \int_x^1 K_2(x,\theta)\gamma_1(\theta,\xi)d\theta w(\xi)d\xi$$
$$- \int_x^1 \int_\xi^1 K_2(x,\theta)\gamma_1(\theta,\xi)d\theta w(\xi)d\xi - \int_x^1 \int_x^\xi K_2(x,\theta)\gamma_2(\theta,\xi)d\theta w(\xi)d\xi.$$

Finally, collecting terms, we obtain

$$\begin{aligned}\left(\mathcal{P}\hat{\mathcal{P}}w\right)(x) =& w(x) + \int_0^x \phi_1(x,\xi)w(\xi)d\xi + \int_x^1 \phi_2(x,\xi)w(\xi)d\xi, \text{ where} \\ \phi_1(x,\xi) =& - M(x)\gamma_1(x,\xi) + K_1(x,\xi)M(\xi)^{-1} - \int_0^\xi K_1(x,\theta)\gamma_2(\theta,\xi)d\theta \\ & - \int_\xi^x K_1(x,\theta)\gamma_1(\theta,\xi)d\theta - \int_x^1 K_2(x,\theta)\gamma_1(\theta,\xi)d\theta, \\ \phi_2(x,\xi) =& - M(x)\gamma_2(x,\xi) + K_2(x,\xi)M(\xi)^{-1} - \int_0^x K_1(x,\theta)\gamma_2(\theta,\xi)d\theta \\ & - \int_x^\xi K_2(x,\theta)\gamma_2(\theta,\xi)d\theta - \int_\xi^1 K_2(x,\theta)\gamma_1(\theta,\xi)d\theta.\end{aligned} \tag{C.10}$$

From Equation (C.9)

$$K_1(x,\xi) = C(x)N_1B(\xi) \quad \text{and} \quad K_2(x,\xi) = -C(x)N_2B(\xi),$$

and from the theorem hypothesis

$$\begin{aligned}\gamma_1(x,\xi) =& M(x)^{-1}C(x)U(x)(I_{4(d+1)} - P)U(\xi)^{-1}B(\xi)M(\xi)^{-1}, \\ \gamma_2(x,\xi) =& - M(x)^{-1}C(x)U(x)PU(\xi)^{-1}B(\xi)M(\xi)^{-1}.\end{aligned}$$

Substituting these values in $\phi_1(x,\xi)$, we obtain

$$\begin{aligned}&\phi_1(x,\xi) \\ &= C(x)\big[- U(x)\left(I_{4(d+1)} - P\right)U(\xi)^{-1} + N_1\big]B(\xi)M(\xi)^{-1} \\ &\quad + C(x)\Bigg[N_1 \int_0^\xi B(\theta)M(\theta)^{-1}C(\theta)U(\theta)d\theta P \\ &\qquad - N_1 \int_\xi^x B(\theta)M(\theta)^{-1}C(\theta)U(\theta)d\theta \left(I_{4(d+1)} - P\right) \\ &\qquad + N_2 \int_x^1 B(\theta)M(\theta)^{-1}C(\theta)U(\theta)d\theta \left(I_{4(d+1)} - P\right)\Bigg]U(\xi)^{-1}B(\xi)M(\xi)^{-1}.\end{aligned} \tag{C.11}$$

Since $U(\theta)$ is the fundamental matrix of $-B(\theta)M(\theta)^{-1}C(\theta)$, from Lemma 6.8

$$B(\theta)M(\theta)^{-1}C(\theta)U(\theta) = -\frac{dU(\theta)}{d\theta} \quad \text{and} \quad U(0) = I_{4(d+1)}. \tag{C.12}$$

Substituting Equation (C.12) into (C.11),

$$\begin{aligned}
\phi_1(x,\xi) =& C(x)\big[- \ U(x)\left(I_{4(d+1)} - P\right)U(\xi)^{-1} + N_1\big]B(\xi)M(\xi)^{-1} \\
&+ C(x)\Bigg[- N_1 \int_0^\xi \frac{dU(\theta)}{d\theta} d\theta P + N_1 \int_\xi^x \frac{dU(\theta)}{d\theta} d\theta \left(I_{4(d+1)} - P\right) \\
&\qquad - N_2 \int_x^1 \frac{dU(\theta)}{d\theta} d\theta \left(I_{4(d+1)} - P\right)\Bigg]U(\xi)^{-1}B(\xi)M(\xi)^{-1} \\
=& C(x)\big[- \ U(x)\left(I_{4(d+1)} - P\right)U(\xi)^{-1} + N_1\big]B(\xi)M(\xi)^{-1} \\
&+ C(x)\Bigg[- N_1 \left(U(\xi) - I_{4(d+1)}\right)P + N_1\left(U(x) - U(\xi)\right)\left(I_{4(d+1)} - P\right) \\
&\qquad - N_2\left(U(1) - U(x)\right)\left(I_{4(d+1)} - P\right)\Bigg]U(\xi)^{-1}B(\xi)M(\xi)^{-1},
\end{aligned}$$

where we have used the fact that $U(0) = I_{4(d+1)}$. Simplifying

$$\begin{aligned}
\phi_1(x,\xi) =& C(x)\big[- U(x)\left(I_{4(d+1)} - P\right)U(\xi)^{-1} + N_1\big]B(\xi)M(\xi)^{-1} \\
&+ C(x)\Bigg[N_1U(x)\left(I_{4(d+1)} - P\right) + N_2U(x)\left(I_{4(d+1)} - P\right) - N_1U(\xi) \\
&+ N_1P - N_2U(1) + N_2U(1)P\Bigg]U(\xi)^{-1}B(\xi)M(\xi)^{-1}
\end{aligned} \tag{C.13}$$

By definition $P = \left(N_1 + N_2U(1)\right)^{-1} N_2U(1)$, thus $N_2U(1) = \left(N_1 + N_2U(1)\right)P$. Hence

$$\begin{aligned}
N_1P - N_2U(1) + N_2U(1)P =& N_1P - \left(N_1 + N_2U(1)\right)P + N_2U(1)P \\
=& N_1P - N_1P - N_2U(1)P + N_2U(1)P \\
=& 0.
\end{aligned} \tag{C.14}$$

Moreover, by definition $N_1 + N_2 = I_{4(d+1)}$. Using this fact and substituting (C.14) into (C.13) produces

$$\phi_1(x,\xi) = C(x)\big[- U(x)\left(I_{4(d+1)} - P\right)U(\xi)^{-1} + N_1\big]B(\xi)M(\xi)^{-1}$$

$$
\begin{aligned}
&+ C(x)\Big[U(x)\left(I_{4(d+1)} - P\right) - N_1 U(\xi)\Big] U(\xi)^{-1} B(\xi) M(\xi)^{-1} \\
=& C(x)\big[-U(x)\left(I_{4(d+1)} - P\right) U(\xi)^{-1} + N_1\big] B(\xi) M(\xi)^{-1} \\
&+ C(x)\Big[U(x)\left(I_{4(d+1)} - P\right) U(\xi)^{-1} - N_1\Big] B(\xi) M(\xi)^{-1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\phi_1(x,\xi) =& C(x)\Big[-U(x)\left(I_{4(d+1)} - P\right) U(\xi)^{-1} + N_1\Big] B(\xi) M(\xi)^{-1} \\
&- C(x)\Big[-U(x)\left(I_{4(d+1)} + P\right) U(\xi)^{-1} - N_1\Big] B(\xi) M(\xi)^{-1} \\
=& 0.
\end{aligned}
\tag{C.15}
$$

Substituting the definitions of K_1 and K_2 from Equation (C.9) and γ_1 and γ_2 from the theorem hypothesis produces

$$
\begin{aligned}
&\phi_2(x,\xi) \\
&= C(x)\big[\quad U(x) P U(\xi)^{-1} - N_2\big] B(\xi) M(\xi)^{-1} \\
&\quad + C(x)\Big[N_1 \int_0^x B(\theta) M(\theta)^{-1} C(\theta) U(\theta) d\theta P \\
&\qquad - N_2 \int_x^\xi B(\theta) M(\theta)^{-1} C(\theta) U(\theta) d\theta P \\
&\qquad + N_2 \int_\xi^1 B(\theta) M(\theta)^{-1} C(\theta) U(\theta) d\theta \left(I_{2(d+1)} - P\right)\Big] U(\xi)^{-1} B(\xi) M(\xi)^{-1}.
\end{aligned}
\tag{C.16}
$$

From Equation (C.12), we have that

$$
B(\theta) M(\theta)^{-1} C(\theta) U(\theta) = -\frac{dU(\theta)}{d\theta} \quad \text{and} \quad U(0) = I_{4(d+1)}.
$$

Thus

$$
\begin{aligned}
\phi_2(x,\xi) =& C(x)\big[\quad U(x) P U(\xi)^{-1} - N_2\big] B(\xi) M(\xi)^{-1} \\
&+ C(x)\Big[-N_1 \int_0^x \frac{dU(\theta)}{d\theta} d\theta P + N_2 \int_x^\xi \frac{dU(\theta)}{d\theta} d\theta P
\end{aligned}
$$

$$
\begin{aligned}
&\qquad - N_2 \int_\xi^1 \frac{dU(\theta)}{d\theta} d\theta \left(I_{2(d+1)} - P\right) \Big] U(\xi)^{-1} B(\xi) M(\xi)^{-1} \\
=& C(x) \big[\quad U(x) P U(\xi)^{-1} - N_2 \big] B(\xi) M(\xi)^{-1} \\
&+ C(x) \Big[- N_1 \left(U(x) - I_{2(d+1)}\right) P + N_2 \left(U(\xi) - U(x)\right) P \\
&\qquad - N_2 \left(U(1) - U(\xi)\right) \left(I_{2(d+1)} - P\right) \Big] U(\xi)^{-1} B(\xi) M(\xi)^{-1},
\end{aligned}
$$

where we have used the fact that $U(0) = I_{4(d+1)}$. Simplifying

$$
\begin{aligned}
\phi_2(x,\xi) =& C(x) \big[U(x) P U(\xi)^{-1} - N_2 \big] B(\xi) M(\xi)^{-1} \\
&+ C(x) \Big[- N_1 U(x) P - N_2 U(x) P + N_2 U(\xi) \\
&+ N_1 P - N_2 U(1) + N_2 U(1) P \Big] U(\xi)^{-1} B(\xi) M(\xi)^{-1}.
\end{aligned}
$$

From Equation (C.14), $N_1 P - N_2 U(1) + N_2 U(1) P = 0$. Additionally $-N_1 - N_2 = -I_{4(d+1)}$. Thus

$$
\begin{aligned}
\phi_2(x,\xi) =& C(x) \big[U(x) P U(\xi)^{-1} - N_2 \big] B(\xi) M(\xi)^{-1} \\
&+ C(x) \Big[- U(x) P + N_2 U(\xi) \Big] U(\xi)^{-1} B(\xi) M(\xi)^{-1} \\
=& C(x) \big[U(x) P U(\xi)^{-1} - N_2 \big] B(\xi) M(\xi)^{-1} \\
&+ C(x) \Big[- U(x) P U(\xi)^{-1} + N_2 \Big] B(\xi) M(\xi)^{-1}.
\end{aligned}
$$

Finally,

$$
\phi_2(x,\xi) = C(x) \Big[U(x) P U(\xi)^{-1} - N_2 - U(x) P U(\xi)^{-1} + N_2 \Big] B(\xi) M(\xi)^{-1} = 0. \tag{C.17}
$$

Substituting Equations (C.15) and (C.17) into (C.10) produces

$$
\left(\mathcal{P}\hat{\mathcal{P}} w\right)(x) = w(x).
$$

Thus, $\mathcal{P}\hat{\mathcal{P}} = \mathcal{I}$. The proof for $\hat{\mathcal{P}}\mathcal{P} = \mathcal{I}$ is similar.

$\square$

APPENDIX D

SOLUTIONS TO PARABOLIC PDES USING SEPARATION OF VARIABLES

For a few types of parabolic PDEs, the solution may be explicitly calculated using a technique known as *separation of variables* [36]. The idea is to represent the solution of the PDE as the product of solutions of two Ordinary Differential Equations (ODEs). We specifically consider the class of PDEs considered in Chapter 5 and use Sturm-Liouville theory [77] to formulate solutions.

Consider the following PDE

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t), \tag{D.1}$$

with boundary conditions of the form

$$\nu_1 w(0,t) + \nu_2 w_x(0,t) = 0 \quad \text{and} \quad \rho_1 w(1,t) + \rho_2 w_x(1,t) = 0. \tag{D.2}$$

The scalars ν_i and ρ_j satisfy

$$|\nu_1| + |\nu_2| > 0 \quad \text{and} \quad |\rho_1| + |\rho_2| > 0.$$

Here, a, b and c are polynomials and $a(x) \geq \alpha > 0$ for all $x \in [0,1]$.

The uniqueness and existence of solutions to such problems has been established in Lemma 5.4. However, using separation of variables, we can establish the structure of solutions and then establish the stability properties. We present the following theorem.

Lemma D.1. *For any initial condition $w_0 \in \mathcal{D}_0(L_2(0,1))$, there exist scalars ω_n and an orthonormal basis ϕ_n of $L_2(0,1)$, $n \in \mathbb{N}$ such that the classical (weak) solution of Equations* (D.1)-(D.2) *is given by*

$$w(x,t) = \sum_{n=0}^{\infty} e^{\omega_n t} \langle w_0, \phi_n \rangle \phi_n(x). \tag{D.3}$$

Moreover,

$$\omega_0 > \omega_1 > \cdots > \omega_n > \cdots \text{ and } \omega_n \to -\infty \text{ as } n \to \infty.$$

Here, the set $\mathcal{D}_0$ *is defined as*

$$\mathcal{D}_0 = \{y \in H^2(0,1) : \quad \nu_1 y(0) + \nu_2 y_x(0) = 0 \textit{ and } \rho_1 y(1) + \rho_2 y_x(1) = 0\}.$$

Proof. We begin by using the ansatz that the solution can be written as

$$w(x,t) = X(x)T(t).$$

Substituting this ansatz into Equation (D.1) produces

$$X(x)T_t(t) = a(x)X_{xx}(x)T(t) + b(x)X_x(x)T(t) + c(x)X(x)T(t),$$

with boundary conditions

$$T(t)\left(\nu_1 X(0) + \nu_2 X_x(0)\right) = 0 \quad \text{and} \quad T(t)\left(\rho_1 X(1) + \rho_2 X_x(1)\right) = 0.$$

Separating spatial and temporal terms

$$\frac{T_t(t)}{T(t)} = \frac{a(x)X_{xx}(x) + b(x)X_x(x) + c(x)X(x)}{X(x)}. \tag{D.4}$$

Since the left hand side is a function of time t only and the right hand side is a function of space x only, in order for (D.4) to be true, the following must hold for some $\lambda \in \mathbb{R}$,

$$\frac{T_t(t)}{T(t)} = \frac{a(x)X_{xx}(x) + b(x)X_x(x) + c(x)X(x)}{X(x)} = -\lambda. \tag{D.5}$$

Thus, we obtain the following ODEs

$$-a(x)X_{xx}(x) - b(x)X_x(x) - c(x)X(x) = \lambda X(x), \tag{D.6}$$

with boundary conditions

$$\nu_1 X(0) + \nu_2 X_x(0) = 0 \quad \text{and} \quad \rho_1 X(1) + \rho_2 X_x(1) = 0, \tag{D.7}$$

and

$$T_t(t) = -\lambda T(t). \tag{D.8}$$

If we define

$$p(x) = e^{\int_0^x \frac{b(\xi)}{a(\xi)} d\xi}, \quad q(x) = -c(x)\frac{p(x)}{a(x)}, \quad \sigma(x) = \frac{p(x)}{a(x)}, \tag{D.9}$$

then Equations (D.6)-(D.7) can be written as

$$(\mathcal{S}X)(x) = -\frac{d}{dx}\left(p(x)\frac{dX(x)}{dx}\right) + q(x)X(x) = \lambda\sigma(x)X(x), \quad X \in \mathcal{D}_0. \tag{D.10}$$

For Definition 5.2, the operator $\mathcal{S}$ is the Sturm-Liouville operator and Equation (D.10) is the Sturm-Liouville equation. Then, form Lemma 5.3, there exist scalars λ_n satisfying

$$\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \text{ and } \lambda_n \to \infty \text{ as } n \to \infty,$$

and functions $X_n = \phi_n \in \mathcal{D}_0$ such that

$$-\frac{d}{dx}\left(p(x)\frac{d\phi_n(x)}{dx}\right) + q(x)\phi_n(x) = \lambda_n\sigma(x)\phi_n(x). \tag{D.11}$$

For each λ_n, the solution of Equation (D.8) can be easily calculated as

$$T_n(t) = A_n e^{-\lambda_n t}, \tag{D.12}$$

for some scalar $A_n \in \mathbb{R}$. Since from the Ansatz we have that

$$w(x,t) = X(x)T(t),$$

for any $n \in \mathbb{N}$, the solution to Equations (D.1)-(D.2) is given by

$$w_n(x,t) = X_n(x)T_n(t) = A_n e^{-\lambda_n t}\phi_n(x).$$

By superposition, the solution of Equations (D.1)-(D.2) is a linear combination of all possible solutions. Thus, there exist scalars $B_n \in \mathbb{R}$ such that

$$w(x,t) = \sum_{n=0}^{\infty} C_n e^{-\lambda_n t}\phi_n(x), \tag{D.13}$$

where $C_n = A_n B_n$. This solution obviously satisfies the boundary conditions (D.2) since $\phi_n \in \mathcal{D}_0$. However, the solution must satisfy $w(x,0) = w_0(x)$. From Lemma 5.3 we have that ϕ_n is an orthonormal basis for $L_2(0,1)$, thus, from [35, Theorem 3.5-2]

$$w_0(x) = \sum_{n=0}^{\infty} \langle w_0, \phi_n \rangle \phi_n(x).$$

Therefore, If we set

$$C_n = \langle w_0, \phi_n \rangle ,$$

then

$$w(x,0) = \sum_{n=0}^{\infty} \langle w_0, \phi_n \rangle \phi_n(x) = w_0(x).$$

Hence, the solution is given by

$$w(x,t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle w_0, \phi_n \rangle e^{-\lambda_n t} \phi_n(x).$$

Finally, setting $\omega_n = -\lambda_n$ produces

$$w(x,t) = \sum_{n=0}^{\infty} e^{\omega_n t} \langle w_0, \phi_n \rangle \phi_n(x).$$

$\square$

From Lemma D.1 we have that

$$\omega_0 > \omega_1 > \cdots > \omega_n > \cdots .$$

Thus, the system represented by Equations (D.1)-(D.2) is exponentially stable if $\omega_0 < 0$. If we can calculate the eigenvalues, we can infer the system's stability properties. Unfortunately, for a system with spatially distributed coefficients, there is no general way of calculating the eigenvalues. However, we can estimate them. For the stability analysis, this will serve as a benchmark against which we can compare the provided methodology. Additionally, this will help us to synthesize static controllers which will serve as a benchmark against which we can compare the performance of the controllers we synthesize. We present the following Lemma.

Lemma D.2. *Given coefficients $a(x)$, $b(x)$ and $c(x)$ of Equation* (D.1)*, define*

$$p(x) = e^{\int_0^x \frac{b(\xi)}{a(\xi)} d\xi}, \quad q(x) = -c(x)\frac{p(x)}{a(x)}, \quad \sigma(x) = \frac{p(x)}{a(x)}.$$

Additionally, let

$$p(x) \geq p_0 > 0, \quad q(x) \geq q_1, \quad \sigma(x) \leq \sigma_1.$$

Then, if $\nu_1\nu_2 \leq 0$ and $\rho_1\rho_2 \geq 0$, we have that

$$\omega_0 \leq -\lambda_0^{cc},$$

where the scalars ω_n define the solution given in Equation (D.3) *and λ_1^{cc} is the first eigenvalue of the following constant coefficient Sturm-Liouville equation*

$$-p_0\frac{d^2z(x)}{dx^2} + q_1z(x) = \lambda\sigma_1 z(x), \quad z \in \mathcal{D}_0.$$

Proof. We begin by commenting that since $a(x) \geq \alpha > 0$, there exists a scalar p_0 such that

$$p(x) = e^{\int_0^x \frac{b(\xi)}{a(\xi)} d\xi} \geq p_0 > 0.$$

Additionally, since $q(x)$ and $\sigma(x)$ are continuous, there exist scalars q_1 and σ_1 such that

$$q(x) \geq q_1, \quad \sigma(x) \leq \sigma_1.$$

Recall from the proof of Lemma D.1 that $\omega_n = -\lambda_n$, where λ_n are the eigenvalues of the following Sturm-Liouville equation

$$-\frac{d}{dx}\left(p(x)\frac{dz(x)}{dx}\right) + q(x)z(x) = \lambda\sigma(x)z(x), \quad z \in \mathcal{D}_0.$$

Using the Rayleigh quotient [93, Chapter 5], the first eigenvalue is given by

$$\lambda_0 = \min_{z\in\mathcal{D}_0} \frac{p(0)y(0)y_x(0) - p(1)y(1)y_x(1) + \int_0^1 \left(p(x)y_x(x)^2 + q(x)y(x)^2\right) dx}{\int_0^1 \sigma(x)y(x)^2 dx}. \quad \text{(D.14)}$$

If $z \in D_0$, then $z \in \hat{\mathcal{D}}_0$, where

$$\hat{\mathcal{D}}_0 = \{y \in H^1(0,1): \quad y_x(0,t) = k_0 y(0,t), \quad y_x(1,t) = k_1 y(1,t)$$
$$w(0,t) = 0 \text{ if } k_0 = 0 \text{ and } w(1,t) = 0 \text{ if } k_1 = 0\},$$

where

$$k_0 = \begin{cases} -\frac{\nu_1}{\nu_2} & \text{if } \nu_2 \neq 0 \\ \\ 0 & \text{if } \nu_2 = 0 \end{cases}, \quad k_1 = \begin{cases} \frac{\rho_1}{\rho_2} & \text{if } \rho_2 \neq 0 \\ \\ 0 & \text{if } \rho_2 = 0 \end{cases},$$

Thus, Equation (D.14) may be written as

$$\lambda_0 = \min_{z \in \hat{\mathcal{D}}_0} \frac{k_0 p(0) y(0)^2 + k_1 p(1) y(1)^2 + \int_0^1 \left(p(x) y_x(x)^2 + q(x) y(x)^2\right) dx}{\int_0^1 \sigma(x) y(x)^2 dx}. \tag{D.15}$$

We assumed that $\nu_1 \nu_2 \leq 0$ and $\rho_1 \rho_2 \geq 0$, thus

$$k_0 \geq 0 \quad \text{and} \quad k_1 \geq 0.$$

Consequently

$$\begin{aligned} &\frac{k_0 p(0) y(0)^2 + k_1 p(1) y(1)^2 + \int_0^1 \left(p(x) y_x(x)^2 + q(x) y(x)^2\right) dx}{\int_0^1 \sigma(x) y(x)^2 dx} \\ \geq\ &\frac{k_0 p_0 y(0)^2 + k_1 p_0 y(1)^2 + \int_0^1 \left(p_0 y_x(x)^2 + q_1 y(x)^2\right) dx}{\int_0^1 \sigma_1 y(x)^2 dx} \end{aligned}$$

Since the right hand side is also a Rayleigh quotient, it follows that

$$\lambda_0 \geq \lambda_0^{cc},$$

where λ_0^{cc} is the first eigenvalue of the following constant coefficient Sturm-Liouville equation

$$-p_0 \frac{d^2 z(x)}{dx^2} + q_1 z(x) = \lambda \sigma_1 z(x), \quad z \in \mathcal{D}_0.$$

Since $\omega_0 = -\lambda_0$, we obtain

$$\omega_0 \leq -\lambda_0^{cc}.$$

$\square$

The advantage of Lemma D.2 is that the eigenvalues of the constant coefficient Sturm-Liouville equation

$$-p_0 \frac{d^2 z(x)}{dx^2} + q_1 z(x) = \lambda \sigma_1 z(x), \quad z \in \mathcal{D}_0,$$

for most boundary conditions, can be calculated analytically. Thus, we can easily obtain an upper bound on ω_1 and thus, wean information on the system stability. Table D.1 summarizes the eigenvalues λ_n^{cc} and eigenfunctions ϕ_n^{cc} for Dirichlet, Neumann, mixed and Robin boundary conditions.

Table D.1. Eigenvalues and normalized eigenfunctions of $-p_0 \frac{d^2 z(x)}{dx^2} + q_1 z(x) = \lambda \sigma_1 z(x)$ with Dirichlet, Neumann, mixed and Robin boundary conditions.

Boundary Conditions	*Eigenvalues* λ_n^{cc}	*Eigenfunctions* ϕ_n^{cc}
Dirichlet		
$w(0) = 0, w(1) = 0$	$\left(p_0 n^2 \pi^2 + q_1\right)/\sigma_1$	$\frac{1}{\sqrt{2}} \sin n\pi x$
Neumann		
$w_x(0) = 0, w_x(1) = 0$	$\left(p_0 n^2 \pi^2 + q_1\right)/\sigma_1$	$\frac{1}{\sqrt{2}} \cos n\pi x$
Mixed		
$w(0) = 0, w_x(1) = 0$	$\left(p_0 (2n-1)^2 \pi^2 + 4q_1\right)/4\sigma_1$	$\frac{1}{\sqrt{2}} \cos((2n-1)\pi/2)x$
Robin		
$w(0) = 0, w(1) + w_x(1) = 0$	$\lambda_n^{cc} \in (\lambda_n^1, \lambda_n^2)$ (see (D.16))	$\frac{1}{\sqrt{2}} \sin \lambda_n^{cc} x$

In Table D.1,

$$\lambda_n^1 = \left(p_0 (2n-1)^2 \pi^2 + 4q_1\right)/4\sigma_1 \quad \text{and} \quad \lambda_n^2 = \left(p_0 n^2 \pi^2 + q_1\right)/\sigma_1. \tag{D.16}$$

APPENDIX E

STABILITY ANALYSIS USING FINITE-DIFFERENCES AND STURM-LIOUVILLE THEORY

In Chapters 5-7 we consider the following two parabolic PDEs:

$$w_t(x,t) = w_{xx}(x,t) + \lambda w(x,t), \text{ and} \tag{E.1}$$

$$\begin{aligned} w_t(x,t) &= \left(x^3 - x^2 + 2\right) w_{xx}(x,t) + \left(3x^2 - 2x\right) w_x(x,t) \\ &\quad + \left(-0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda\right) w(x,t), \end{aligned} \tag{E.2}$$

where λ is a scalar which may be chosen freely. We consider the following boundary conditions for these two equations:

$$\text{Dirichlet:} = w(0) = 0, \quad w(1) = 0, \tag{E.3}$$

$$\text{Neumann:} = w_x(0) = 0, \quad w_x(1) = 0, \tag{E.4}$$

$$\text{Mixed:} = w(0) = 0, \quad w_x(1) = 0, \tag{E.5}$$

$$\text{Robin:} = w(0) = 0, \quad w(1) + w_x(1) = 0. \tag{E.6}$$

Using Lemma D.1 we may analytically compute the interval in which the scalar λ must lie such that Equation (E.1) is exponentially stable. However, for Equation (E.2), the eigenvalues can not be computed analytically, in which case, we may approximate the interval in which λ must lie for exponential stability using Lemma D.2 or finite-differences.

We begin first by considering Equation (E.1) with boundary conditions (E.3)-(E.6). This equation corresponds to

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t)$$

with

$$a(x) = 1, \quad b(x) = 0, \quad c(x) = \lambda.$$

If we let

$$p(x) = e^{\int_0^x \frac{b(\xi)}{a(\xi)} d\xi}, \quad q(x) = -c(x)\frac{p(x)}{a(x)}, \quad \sigma(x) = \frac{p(x)}{a(x)},$$

then, we get

$$p(x) = p_0 = 1, \quad q(x) = q_1 = -\lambda, \quad \sigma(x) = \sigma_1 = 1. \tag{E.7}$$

Then, by Lemma D.1, the solution of Equation (E.1) is given by

$$w(x,t) = \sum_{n=0}^{\infty} e^{\omega_n t} \langle w_0, \phi_n \rangle \phi_n(x),$$

where w_0 is an appropriately chosen initial condition and $\omega_n = -\lambda_n^{cc}$, where λ_n^{cc} and ϕ_n are the eigenvalues and normalized eigenfunctions, respectively, of the following constant coefficient Sturm-Liouville equation

$$-p_0 \frac{d^2 z(x)}{dx^2} + q_1 z(x) = \lambda^{cc} \sigma_1 z(x).$$

Using the values in (E.7) and Table D.1, the solution of Equation (E.1) with Dirichlet boundary conditions (E.3) is given by

$$w(x,t) = \sum_{n=0}^{\infty} e^{\left(\lambda - n^2\pi^2\right)t} \langle w_0, \phi_n \rangle \phi_n(x), \tag{E.8}$$

where $\phi_n(x) = \frac{1}{\sqrt{2}} \sin n\pi x$. Therefore, for Dirichlet boundary conditions, Equation (E.1) is stable for $\lambda \in [0, \pi^2)$. Similarly, the solution of Equation (E.1) for Neumann and mixed boundary conditions, respectively, is

$$w(x,t) = \sum_{n=0}^{\infty} e^{\left(\lambda - n^2\pi^2\right)t} \langle w_0, \phi_n \rangle \phi_n(x), \tag{E.9}$$

where $\phi_n(x) = \frac{1}{\sqrt{2}} \cos n\pi x$, and

$$w(x,t) = \sum_{n=1}^{\infty} e^{\left(\lambda - (2n-1)^2\pi^2/4\right)t} \langle w_0, \phi_n \rangle \phi_n(x), \tag{E.10}$$

where $\phi_n(x) = \frac{1}{\sqrt{2}} \sin n\pi x$. From Equation (E.9), for Neumann boundary condition, the system governed by Equation (E.1) is stable for $\lambda \in [0, \pi^2)$. Similarly, from Equation (E.10), for mixed boundary condition, the system governed by Equation (E.1) is stable for $\lambda \in [0, \pi^2/4)$.

Finally, for the Robin boundary conditions, using (E.7) and Table D.1, we have that

$$\lambda - n^2\pi^2 \le -\lambda_n^{cc} = \omega_n \le \lambda - \frac{(2n-1)^2\pi^2}{4}.$$

Thus, the solution of Equation (E.1) with Robin boundary conditions satisfies

$$w(x,t) = \sum_{n=1}^{\infty} e^{\omega_n t} \langle w_0, \phi_n \rangle \phi_n(x), \tag{E.11}$$

where $\phi_n(x) = \frac{1}{\sqrt{2}} \sin \lambda_n^{cc} x$. Since

$$\lambda - n^2\pi^2 \le \omega_n \le \lambda - \frac{(2n-1)^2\pi^2}{4},$$

the solution of Equation (E.1) with Robin boundary conditions is exponentially stable for $\lambda \in [0, \pi^2/4)$. However, this bound on λ is conservative. Thus, we can compliment it by calculating the approximate solution using finite-differences. The state norm $\|w(\cdot,t)\|$ is presented in Figure E.1. It is evident from the figure that Equation (E.1) with Robin boundary conditions is stable for $\lambda < 4.12$.

The stability margins for λ in Equation (E.1) with various boundary conditions is presented in Table E.1.

As stated earlier, analytical solutions for Equation (E.2) can not be calculated. Thus, we rely solely on finite-differences to approximate the upper bounds for the parameter λ so that the system is stable. Figures E.2-E.5 illustrate the state norm $\|w(\cdot,t)\|$ of Equation (E.2) with various boundary conditions.

The stability margins for λ in Equation (E.2) with various boundary conditions is presented in Table E.2.

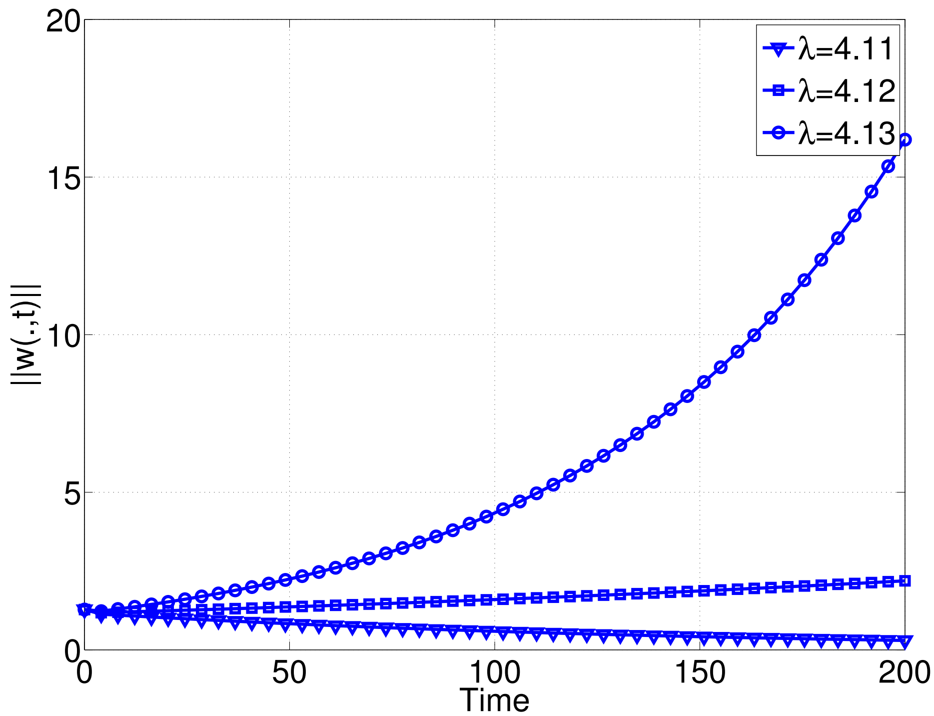


Figure E.1. State norm $\|w(\cdot.t)\|$ of Equation (E.1) with Robin boundary conditions for different λ.

Table E.1. Stability margins for $\lambda > 0$ for $w_t = w_{xx} + \lambda w$ with Dirichlet, Neumann, mixed and Robin boundary conditions.

Boundary Conditions	Stability margin for $\lambda > 0$
Dirichlet	
$w(0) = 0, w(1) = 0$	$\lambda < \pi^2$
Neumann	
$w_x(0) = 0, w_x(1) = 0$	$\lambda < 0$
Mixed	
$w(0) = 0, w_x(1) = 0$	$\lambda < \pi^2/4$
Robin	
$w(0) = 0, w(1) + w_x(1) = 0$	$\lambda < 4.12$

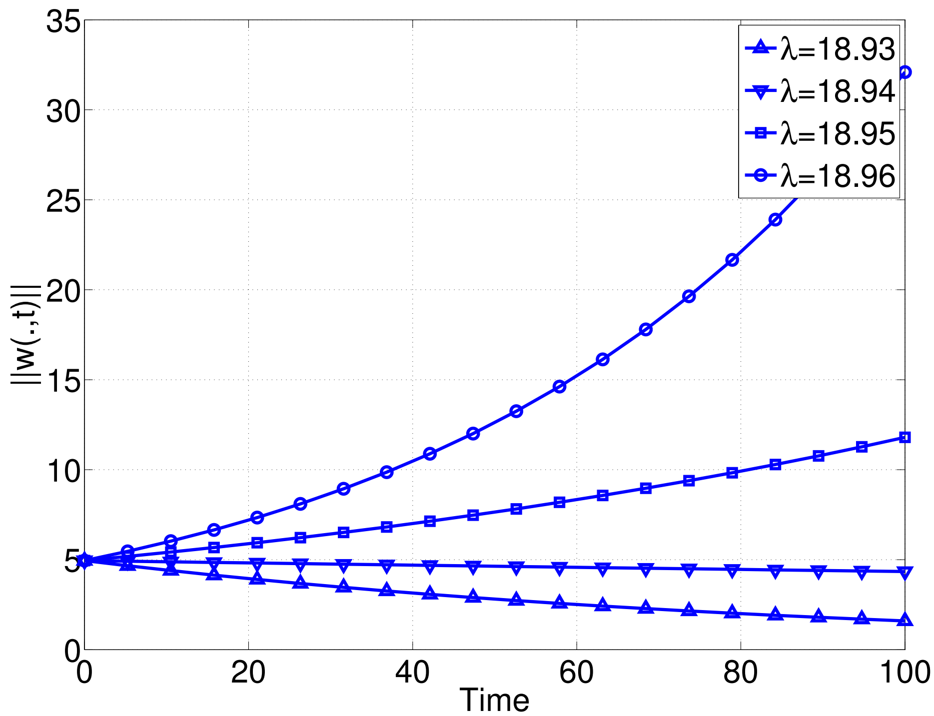


Figure E.2. State norm $\|w(\cdot.t)\|$ of Equation (E.2) with Dirichlet boundary conditions $w(0,t) = w(1,t) = 0$ for different λ.

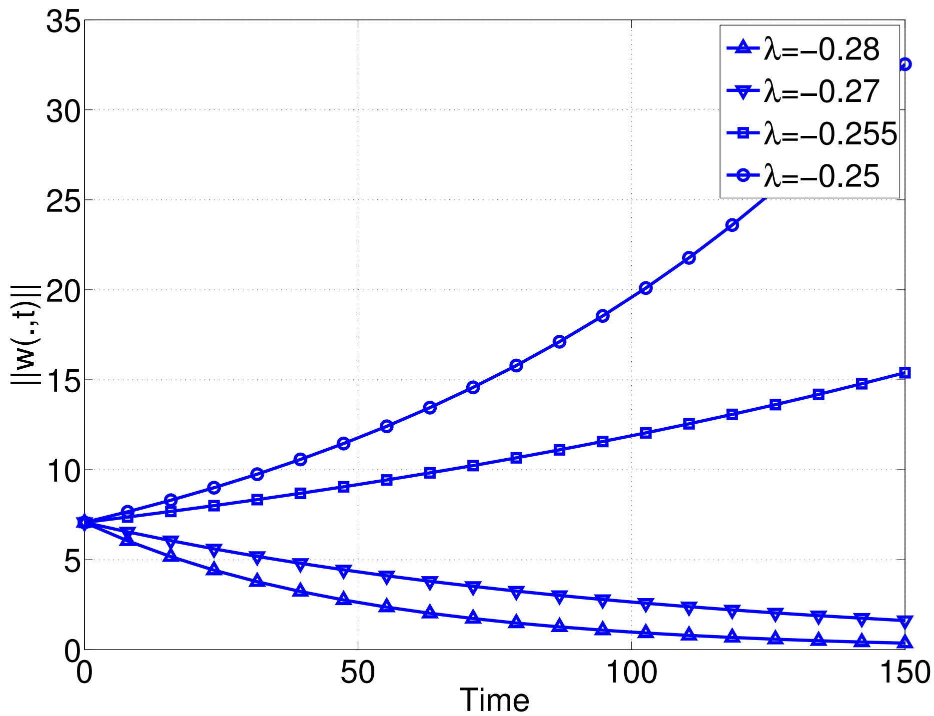


Figure E.3. State norm $\|w(\cdot.t)\|$ of Equation (E.2) with Neumann boundary conditions $w_x(0,t) = w_x(1,t) = 0$ for different λ.

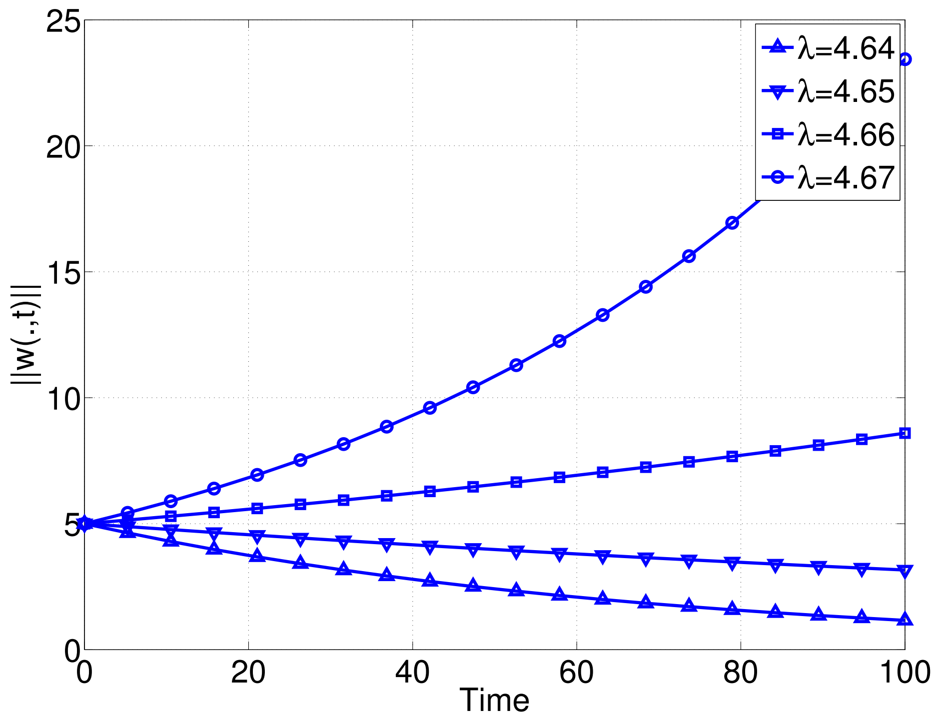


Figure E.4. State norm $\|w(\cdot.t)\|$ of Equation (E.2) with mixed boundary conditions $w(0,t) = w_x(1,t) = 0$ for different λ.

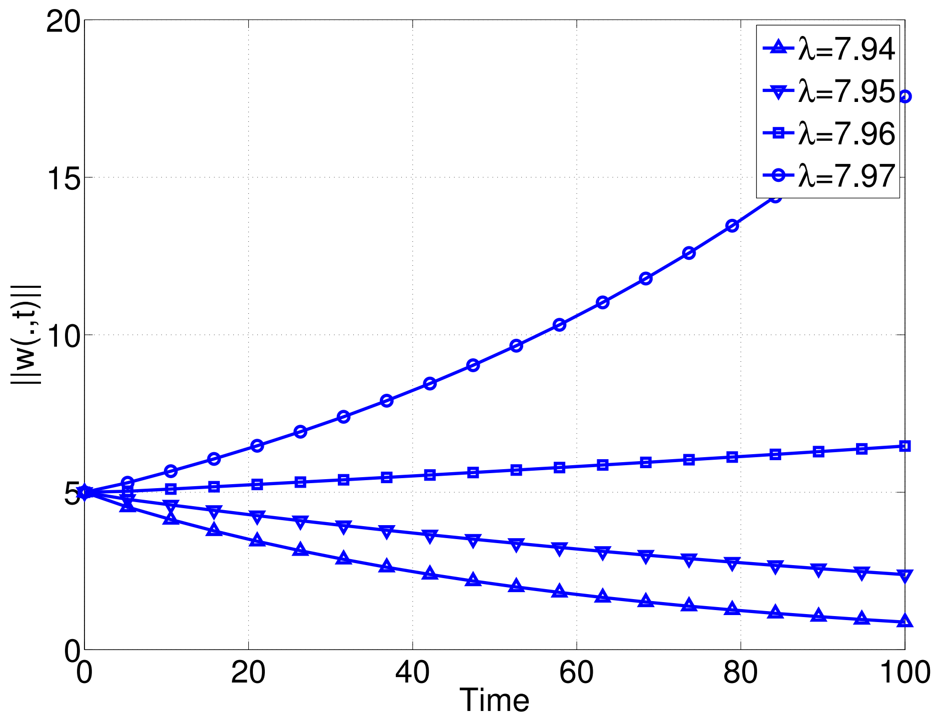


Figure E.5. State norm $\|w(\cdot.t)\|$ of Equation (E.2) with Robin boundary conditions $w(0,t) = w(1,t) + w_x(1,t) = 0$ for different λ.

Table E.2. Stability margins for Equation (E.2) with Dirichlet, Neumann, mixed and Robin boundary conditions.

Boundary Conditions	Stability margin for $\lambda > 0$
Dirichlet	
$w(0) = 0, w(1) = 0$	$\lambda < 18.95$
Neumann	
$w_x(0) = 0, w_x(1) = 0$	$\lambda < -0.255$
Mixed	
$w(0) = 0, w_x(1) = 0$	$\lambda < 4.66$
Robin	
$w(0) = 0, w(1) + w_x(1) = 0$	$\lambda < 7.96$

BIBLIOGRAPHY

[1] "U.S. Energy Information Administration." `http://www.eia.gov/`.

[2] "Statistical Review of World Energy 2013." `http://www.bp.com/en/global/corporate/about-bp/energy-economics/statistical-review-of-world-energy-2013.html`.

[3] "EFDA-JET, the world's largest nuclear fusion research experiment [Online]." `http://www.jet.efda.org/`.

[4] "ITER organization." `http://www.iter.org/`.

[5] R. Serway and J. Jewett, *Physics for scientists and engineers*. Cengage Learning, 2013.

[6] J. Wesson and D. Campbell, *Tokamaks*. Oxford University Press, USA, 2004.

[7] A. Pironti and M. Walker, "Fusion, tokamaks, and plasma control: An introduction and tutorial," *IEEE Control Systems*, vol. 25, pp. 30–43, 2005.

[8] E. A. Azizov, "Tokamaks: from A. D. sakharov to the present (the 60-year history of tokamaks)," *Physics-Uspekhi*, vol. 55, p. 190, 2012.

[9] P. H. Rebut, R. J. Bickerton, and B. E. Keen, "The Joint European Torus: installation, first results and prospects," *Nuclear fusion*, vol. 25, p. 1011, 1985.

[10] F. Parlange and C. Leloup, "Tore Supra tokamak," in *Electricity uses*, 1997.

[11] R. Hastie, "Sawtooth instability in tokamak plasmas," *Astrophysics and space science*, vol. 256, pp. 177–204, 1997.

[12] O. Sauter *et al.*, "Control of neoclassical tearing modes by sawtooth control," *Physical review letters*, vol. 88, no. 10, p. 105001, 2002.

[13] M. L. Walker *et al.*, "Emerging applications in tokamak plasma control," *IEEE Control Systems Magazine*, vol. 26, pp. 35–63, 2006.

[14] F. D. Halpern, G. Bateman, and A. H. Kritz, "Integrated simulations of saturated neoclassical tearing modes in DIII-D, Joint European Torus, and ITER plasmas," *Physics of Plasmas (1994-present)*, vol. 13, p. 062510, 2006.

[15] M. Ariola and A. Pironti, "Plasma shape control for the JET tokamak: An optimal output regulation approach," *IEEE Control Systems*, vol. 25, pp. 65–75, 2005.

[16] Y. Liang, H. R. Koslowski, P. R. Thomas, E. Nardon, B. Alper, P. Andrew, Y. Andrew, G. Arnoux, Y. Baranov, M. Bécoulet, *et al.*, "Active control of type-i edge-localized modes with n= 1 perturbation fields in the JET tokamak," *Physical review letters*, vol. 98, p. 265004, 2007.

[17] J. R. Ferron, P. Gohil, C. M. Greenfield, J. Lohr, T. C. Luce, M. A. Makowski, D. Mazon, M. Murakami, C. C. Petty, P. A. Politzer, *et al.*, "Feedback control of the safety factor profile evolution during formation of an advanced tokamak discharge," *Nuclear fusion*, vol. 46, p. L13, 2006.

[18] F. B. Argomedo, E. Witrant, C. Prieur, S. Brémond, R. Nouailletas, J.-F. Artaud, *et al.*, "Lyapunov-based infinite-dimensional control of the safety factor profile in a tokamak plasma," *Nuclear Fusion*, vol. 53, 2013.

[19] E. A. Lazarus, G. A. Navratil, C. M. Greenfield, E. J. Strait, M. E. Austin, K. H. Burrell, T. A. Casper, D. R. Baker, J. C. DeBoo, E. J. Doyle, *et al.*, "Higher fusion power gain with current and pressure profile control in strongly shaped DIII-D tokamak plasmas," *Physical review letters*, vol. 77, p. 2714, 1996.

[20] C.-M. Fransson, B. Lennartson, C. Breitholtz, A. Bondeson, and Y. Liu, "Feedback stabilization of nonaxisymmetric resistive wall modes in tokamaks. II. Control analysis," *Physics of Plasmas (1994-present)*, vol. 7, pp. 4143–4151, 2000.

[21] T. S. Taylor, "Physics of advanced tokamaks," *Plasma Physics and Controlled Fusion*, vol. 39, p. B47, 1997.

[22] JET, "Fusion energy production from a deuterium-tritium plasma in the JET tokamak," *Nuclear Fusion*, vol. 32, p. 187, 1992.

[23] V. S. Chan, C. M. Greenfield, L. L. Lao, T. C. Luce, C. C. Petty, G. M. Staebler, and D.-D. Team, "DIII-D advanced tokamak research overview," *Nuclear Fusion*, vol. 40, p. 1137, 2000.

[24] D. Moreau, D. Mazon, M. Ariola, G. De Tommasi, A. Boboc, M. Brix, J. Brzozowski, V. Cocilovo, R. Felton, N. Hawkes, *et al.*, "Real-time profile control for advanced tokamak operation on JET," in *Proceedings of the 5th IAEA Technical Meeting on Steady State Operations of Magnetic Fusion Devices (Daejeon, Republic of Korea*, 2007.

[25] D. Moreau, F. Crisanti, X. Litaudon, D. Mazon, P. De Vries, R. Felton, E. Joffrin, L. Laborde, M. Lennholm, A. Murari, *et al.*, "Real-time control of the q-profile in JET for steady state advanced tokamak operation," *Nuclear fusion*, vol. 43, p. 870, 2003.

[26] C. M. Greenfield, J. C. DeBoo, T. C. Luce, B. W. Stallard, E. J. Synakowski, L. R. Baylor, K. H. Burrell, T. A. Casper, E. J. Doyle, D. R. Ernst, *et al.*, "Understanding and control of transport in advanced tokamak regimes in DIII-D," *Physics of Plasmas (1994-present)*, vol. 7, pp. 1959–1967, 2000.

[27] M. Okabayashi, J. Bialek, M. S. Chance, M. S. Chu, E. D. Fredrickson, A. M. Garofalo, M. Gryaznevich, R. E. Hatcher, T. H. Jensen, L. C. Johnson, *et al.*, "Active feedback stabilization of the resistive wall mode on the DIII-D device," *Physics of Plasmas (1994-present)*, vol. 8, pp. 2071–2082, 2001.

[28] K. Miyamoto, *Plasma physics and controlled nuclear fusion*, vol. 38. Springer, 2006.

[29] J. P. Freidberg, "Ideal magnetohydrodynamic theory of magnetic fusion systems," *Reviews of Modern Physics*, vol. 54, p. 801, 1982.

[30] J. Freidberg, *Plasma Physics and Fusion Energy*. Cambridge University Press, 2007.

[31] L. Evans, "Partial differential equations," *Graduate Studies in Mathematics*, vol. 19, 1998.

[32] E. C. Zachmanoglou and D. W. Thoe, *Introduction to partial differential equations with applications.* Courier Dover Publications, 2012.

[33] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 198. Springer New York, 1983.

[34] J. Hadamard, *Lectures on Cauchy's problem in linear partial differential equations.* Courier Dover Publications, 2003.

[35] E. Kreyszig, *Introductory functional analysis with applications*, vol. 21. Wiley, 1989.

[36] M. Krstic and A. Smyshlyaev, *Boundary control of PDEs: A course on backstepping designs*, vol. 16. Society for Industrial Mathematics, 2008.

[37] P. D. Christofides and P. Daoutidis, "Finite-dimensional control of parabolic PDE systems using approximate inertial manifolds," in *Proceedings of the 36th IEEE Conference on Decision and Control*, vol. 2, pp. 1068–1073, 1997.

[38] N. H. El-Farra, A. Armaou, and P. D. Christofides, "Analysis and control of parabolic PDE systems with input constraints," *Automatica*, vol. 39, pp. 715–725, 2003.

[39] M. Krstic and A. Smyshlyaev, "Adaptive boundary control for unstable parabolic PDEs, Part I: Lyapunov design," *IEEE Transactions on Automatic Control*, vol. 53, pp. 1575–1591, 2008.

[40] A. Smyshlyaev and M. Krstic, "Backstepping observers for a class of parabolic PDEs," *Systems and Control Letters*, vol. 54, pp. 613–625, 2005.

[41] S. Dubljevic and P. D. Christofides, "Predictive control of parabolic PDEs with boundary control actuation," *Chemical Engineering Science*, vol. 61, pp. 6239–6248, 2006.

[42] M. Krstic, A. A. Siranosian, and A. Smyshlyaev, "Backstepping boundary controllers and observers for the slender Timoshenko beam: Part I-Design," in *Proceedings of the American Control Conference*, pp. 2412–2417, 2006.

[43] S. Boyd and L. Vandenberghe, *Convex optimization.* Cambridge University Press, 2004.

[44] Y. Nesterov, A. Nemirovskii, and Y. Ye, *Interior-point polynomial algorithms in convex programming*, vol. 13. SIAM, 1994.

[45] R. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*, vol. 21. Springer, 1995.

[46] P. Parrilo, *Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization.* PhD thesis, California Institute of Technology, 2000.

[47] E. Witrant, E. Joffrin, S. Brémond, G. Giruzzi, D. Mazon, O. Barana, and P. Moreau, "A control-oriented model of the current profile in tokamak plasma," *Plasma Physics and Controlled Fusion*, vol. 49, p. 1075, 2007.

[48] W. Rudin, "Functional analysis. International series in pure and applied mathematics," 1991.

[49] W. Karush, "Minima of functions of several variables with inequalities as side constraints," Master's thesis, Deptartment of Mathematics, University of Chicago, 1939.

[50] H. W. Kuhn and A. W. Tucker, "Nonlinear programming," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pp. 481–492, 1951.

[51] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM review*, vol. 38, pp. 49–95, 1996.

[52] G. Dullerud and F. Paganini, *A course in robust control theory: A convex approach*. Springer Verlag, 2000.

[53] F. Alizadeh, J. A. Haeberly, and M. L. Overton, "Primal-dual interior-point methods for semidefinite programming: Convergence rates, stability and numerical results," *SIAM Journal on Optimization*, vol. 8, pp. 746–768, 1998.

[54] S. Mehrotra, "On the implementation of a primal-dual interior point method," *SIAM Journal on optimization*, vol. 2, pp. 575–601, 1992.

[55] L. Blum, *Complexity and real computation*. Springer Verlag, 1998.

[56] V. Powers and T. Wörmann, "An algorithm for sums of squares of real polynomials," *Journal of pure and applied algebra*, vol. 127, pp. 99–104, 1998.

[57] G. Chesi, A. Tesi, A. Vicino, and R. Genesio, "On convexification of some minimum distance problems," in *Proceedings of European control conference*, 1999.

[58] J. B. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM Journal on Optimization*, vol. 11, pp. 796–817, 2001.

[59] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: A general purpose sum of squares programming solver," in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 1, pp. 741–746, 2002.

[60] D. Henrion and J.-B. Lasserre, "Gloptipoly: Global optimization over polynomials with MATLAB and SeDuMi," *ACM Transactions on Mathematical Software (TOMS)*, vol. 29, pp. 165–194, 2003.

[61] G. Stengle, "A Nullstellensatz and a Positivstellensatz in semialgebraic geometry," *Mathematische Annalen*, vol. 207, pp. 87–97, 1974.

[62] K. Schmüdgen, "The K-moment problem for compact semi-algebraic sets," *Mathematische Annalen*, vol. 289, pp. 203–206, 1991.

[63] M. Putinar, "Positive polynomials on compact semi-algebraic sets," *Indiana University Mathematics Journal*, vol. 42, pp. 969–984, 1993.

[64] M. Laurent, "Sums of squares, moment matrices and optimization over polynomials," in *Emerging applications of algebraic geometry*, pp. 157–270, Springer, 2009.

[65] C. Scheiderer, "Positivity and sums of squares: A guide to recent results," in *Emerging applications of algebraic geometry*, pp. 271–324, 2009.

[66] M. Murakami *et al.*, "Progress toward fully noninductive, high beta conditions in DIII-D," *Physics of Plasmas (1994-present)*, vol. 13, p. 056106, 2006.

[67] J. Blum, *Numerical Simulation and Optimal Control in Plasma Physics.* Gauthier-Villars, John Wiley and Sons, 1989.

[68] F. B. Argomedo, C. Prieur, E. Witrant, and S. Brémond, "A strict control Lyapunov function for a diffusion equation with time-varying distributed coefficients," *IEEE Transactions on Automatic Control*, vol. 58, pp. 290–303, 2012.

[69] S. P. Hirshman, R. J. Hawryluk, and B. Birge, "Neoclassical conductivity of a tokamak plasma," *Nuclear Fusion*, vol. 17, 1977.

[70] S. B. Krivit, J. H. Lehr, and T. B. Kingery, *Nuclear energy encyclopedia: science, technology, and applications*, vol. 4. John Wiley & Sons, 2011.

[71] S. P. Hirshman, "Finite-aspect-ratio effects on the bootstrap current in tokamaks," *Physics of Fluids (1958-1988)*, vol. 31, pp. 3150–3152, 1988.

[72] G. T. Hoang and R. V. Budny, "The bootstrap fraction in TFTR," in *Proceedings of the twelfth topical conference on radio frequency power in plasmas*, vol. 403, pp. 235–238, 1997.

[73] F. Imbeaux, *Etude de la propagation et de l'absorption de l'onde hybride dans un plasma de tokamak par tomographie X haute énergie.* PhD thesis, Université Paris XI Orsay, 1999.

[74] O. Barana, D. Mazon, G. Caulier, D. Gamier, M. Jouve, L. Laborde, and Y. Peysson, "Real-time determination of suprathermal electrons local emission profile from hard X-ray measurements in Tore Supra," in *Proccedings of the 14th IEEE-NPSS Real Time Conference, 2005*, p. 5, 2005.

[75] M. Goniche *et al.*, "Lower hybrid current drive efficiency on Tore Supra and JET," in *Radio Frequency Power in Plasmas*, vol. 787, pp. 307–310, 2005.

[76] D. Moreau, D. Mazon, M. Ariola, G. Tommasi, L. Laborde, F. Piccolo, F. Sartori, T. Tala, L. Zabeo, A. Boboc, *et al.*, "A two-time-scale dynamic-model approach for magnetic and kinetic profile control in advanced tokamak scenarios on JET," *Nuclear Fusion*, vol. 48, p. 106001, 2008.

[77] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations.* Springer, 2011.

[78] C. Delattre, D. Dochain, and J. Winkin, "Sturm-Liouville systems are Riesz-spectral systems," *International Journal of Applied Mathematics and Computer Science*, vol. 13, pp. 481–484, 2003.

[79] M. M. Peet and A. Papachristodoulou, "Using polynomial semi-separable kernels to construct infinite-dimensional Lyapunov functions," in *Proceedings of 47th IEEE Conference on Decision and Control*, pp. 847–852, 2008.

[80] M. M. Peet, "LMI parametrization of Lyapunov functions for infinite-dimensional systems: A toolbox," *Submited to American Control Conference (ACC), 2014. Available on control.asu.edu.*

[81] K. Gu, V. Kharitonov, and J. Chen, *Stability of time-delay systems.* Birkhauser, 2003.

[82] M. M. Peet and A. Papachristodoulou, "Inverses of positive linear operators and state feedback design for timedelay systems," in *Proccedings of 8th IFAC Workshop on Time-Delay Systems*, 2009.

[83] M. M. Peet, "Full-state feedback of delayed systems using SOS: A new theory of duality," in *Proceedings of 11th IFAC Workshop on Time-Delay Systems*, 2013.

[84] I. Gohberg and M. A. Kaashoek, "Time varying linear systems with boundary conditions and integral operators. I. The transfer operator and its properties," *Integral equations and Operator theory*, vol. 7, pp. 325–391, 1984.

[85] J. L. Daleckij, M. G. Krejn, and S. Smith, *Stability of solutions of differential equations in Banach space*, vol. 43. American Mathematical Society Providence, RI, 1974.

[86] L. Eriksson, C. Fourment, V. Fuchs, X. Litaudon, C. Challis, F. Crisanti, B. Esposito, X. Garbet, C. Giroud, N. Hawkes, *et al.*, "Discharges in the JET tokamak where the safety factor profile is identified as the critical factor for triggering internal transport barriers," *Physical review letters*, vol. 88, p. 145001, 2002.

[87] A. Gahlawat, M. Peet, and E. Witrant, "Control and verification of the safety-factor profile in tokamaks using sum-of-squares polynomials," in *Preprints of the 18th IFAC World Congress*, 2011.

[88] B. J. Green, "ITER: burning plasma physics experiment," *Plasma physics and controlled fusion*, vol. 45, p. 687, 2003.

[89] M. Kikuchi, "Steady state tokamak reactor based on the bootstrap current," *Nuclear fusion*, vol. 30, p. 265, 1990.

[90] K. C. Shaing, A. Y. Aydemir, Y. R. Lin-Liu, and R. L. Miller, "Steady state tokamak equilibria without current drive," *Physical review letters*, vol. 79, pp. 3652–3655, 1997.

[91] F. A. Bribiesca, C. Prieur, E. Witrant, and S. Brémond, "A strict control lyapunov function for a diffusion equation with time-varying distributed coefficients," *IEEE Transactions on Automatic Control*, vol. 58, pp. 290–303, 2013.

[92] G. Hardy, J. Littlewood, and G. Polya, *Inequalities.* Cambridge University Press, 1988.

[93] Y. Egorov and V. Kondratiev, *On spectral theory of elliptic operators, volume 89 of Operator Theory: Advances and Applications.* Birkhäuser Verlag, Basel, 1996.